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REFERENCE DEPENDENCE AND MARKET PARTICIPATION*

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Abstract

This paper finds optimal portfolios for the reference-dependent preferences by Kőszegi and Rabin, with piecewise linear gain-loss utility, in a one-period model with a safe and a risky asset. If the return of the risky asset is highly dispersed relative to its potential gains, two personal equilibria arise, one of them including risky investments, the other one only safe holdings. In the same circumstances, the risky personal equilibrium entails market participation that decreases with loss aversion and gain-loss sensitivity, whereas the preferred personal equilibrium is sensitive to market and preference parameters. Relevant market parameters are not the expected return and standard deviation, but rather the ratio of expected gains to losses and the Gini index of the return.

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1 Introduction

Standard portfolio theory imagines investors as utility maximizers, unencumbered by personal past references, who are sensitive to consumption and wealth outcomes alone. An important implication is that every such investor should participate, however little, in investments carrying a positive risk premium: using the words of Huang and Litzenberger (1988), "An individual who is risk averse and who strictly prefers more to less will undertake risky investments if and only if the rate of return on at least one risky asset exceeds the risk-free interest rate."

In reality, less than half of households participates in the stock market (Guiso, Haliassos, and Jappelli, 2002; Vissing-Jorgensen, 2003), and the prospect theory of Kahneman and Tversky (1979) recognizes reference dependence as a major determinant of preferences, though it remains silent on the origin of such references. Filling this void, Shalev (2000) argues that "Reference points emerge as expressions of anticipation which are fulfilled", and Kőszegi and Rabin (2006) stipulate that reference points are "rational expectations held in the recent past about outcomes".

This paper solves the one-period portfolio choice problem for an investor with the reference-dependent preferences of Kőszegi and Rabin (2006), and finds that its solution supports two competing *personal equilibria*—expectations about one's choice that make such choice optimal. One personal equilibrium involves a mix of risky and safe investments, and is a variant of the familiar utility-maximizing portfolio of Markowitz (1952). The additional equilibrium involves only safe investments, and offers an explanation for non-participation in the market based on reference-dependent preferences: for an investor expecting to take risk, it is optimal to do so to the extent specified by the risky equilibrium. Yet, for an investor with the same preferences but with the expectation to hold safe assets, it is optimal to forgo risky assets completely.

The reference-dependent preferences of Kőszegi and Rabin (2006) prescribe that the overall value of a payoff results from its standard expected utility plus a further component that measures the satisfaction or disappointment of the payoff in comparison to another reference payoff. The comparison is performed by averaging a gain-loss function over all possible payoff-reference pairs, each of them weighted by its respective probability. This gain-loss function is "kinked" at the origin, with a steeper slope for losses than for gains to reflect *loss aversion*—"losses loom larger than gains" (Kahneman and Tversky, 1979).

While Kőszegi and Rabin (2006, 2007, 2008, 2009) develop reference-dependent preferences at increasing levels of generality, their implications for portfolio choice have hitherto remained unexplored, and this paper starts to fill this gap. The central difference from previous models of reference dependence (cf. Bernard and Ghossoub, 2010; He and Zhou, 2011) is that the reference point is endogenous, and therefore needs to be identified as part of the optimization.

Two main issues arise: First, as optimal choices depend on the reference as well as the utility, multiple optimal portfolios may exist, even with a strictly concave utility function. Second, a reference must be a personal equilibrium, in that it needs to be the optimal payoff for those who adopt it as a reference, e.g., investors cannot increase their utilities by adopting unrealistically low references as to surprise themselves with brilliant results. In mathematical terms, a fixed-point problem appears.

We characterize all the personal equilibria in a one-period model of portfolio choice: First, we solve the optimization problem for an arbitrary reference. Then, we identify those references that reproduce themselves as optimal payoffs. Among them, we further determine the preferred personal equilibrium—the ideal reference that an investor unencumbered by a past reference would choose.

We find that the statistical attributes of asset returns that separate the participation and non-participation regimes are not the first two moments typical of mean-variance analysis, but rather the gain-loss ratio and the Gini index. The ratio of expected gains to expected losses varies from zero for a sure loss to infinity for a sure gain, and already appears in the work of Bernardo and Ledoit (2000), who investigate the asset pricing restrictions for the gain-loss ratio, in analogy to the analysis of Hansen and Jagannathan (1991) on Sharpe ratios. Reference-dependent preferences make this ratio prominent as a result of loss-aversion, and offer theoretical support for its use in asset pricing.

The role of the Gini index as relevant measure of dispersion stems from the definition of reference-dependent utility, which contributes to preferences through the expected gain-loss of payoff-reference outcomes. As payoff and reference are sampled independently in this definition, and our gain-loss function is piecewise linear, it follows that the expected gain-loss of a personal equilibrium is a function of the the mean-absolute difference[1] of such payoff, and the Gini index coincides with the mean-absolute difference divided by twice the mean.

The Gini index also affects the incidence of market participation and non-participation. As the main result (Theorem 3.4 below) shows, and in contrast to the initial quote from Huang and Litzenberger (1988), a risky asset with high gain-loss ratio (hence expected return) is not sufficient to guarantee participation, which in turn requires that the return's Gini index is sufficiently low. Even then, participation and non-participation may coexist as two competing personal equilibria, each of which is optimally chosen by those who already take it as a reference. Of these two personal equilibria, which one is preferred depends on parameter values, as we demonstrate by studying a model in detail.

The paper proceeds as follows: Section 2 describes the model and formulates the reference-dependent optimization problem; Section 3 states the main results and discusses their significance; Section 4 focuses on a model with exponential utility and Gaussian returns, calculating in detail optimal portfolios and their performance; Section 5 discusses further implications and concludes. All proofs are in the appendix.

2 Model

Consider a one-period model in which an investor trades at time 0 and evaluates the payoff at time T, when its return is revealed. The market includes a safe asset with constant interest rate $r \geq -1$, so that the gross return $1+r$ is non-negative, and a risky asset with *excess return* described by a random variable X defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

At time 0, no information is available (i.e., the σ-algebra $\mathscr{F}_0 = \{\varnothing, \Omega\}$ is trivial), while at time T all information is revealed (i.e., $\mathscr{F}_T = \mathscr{F}$). Thus, a self-financing portfolio is described by the initial capital w_0 and the exposure $\phi \in \mathbb{R}$ to the risky asset at time 0. Its terminal value is

$$V_T^{\phi} = \widetilde{w_0} + \phi X,$$

where $\widetilde{w_0} := w_0 (1 + r)$ denotes the compounded initial wealth.

The next assumption ensures that the utility function is smooth and the usual utility maximization problem is well-posed.

Assumption 2.1.

(i) *The risky asset return X is integrable (i.e., $\mathbb{E}[|X|] < +\infty$), arbitrage-free*[2] *(i.e., $\mathbb{P}\{X > 0\} > 0$ and $\mathbb{P}\{X < 0\} > 0$), and has a bounded density $f(\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}$.*[3]

(ii) *The* utility *function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing, concave, and continuously differentiable. Moreover, there exists $\epsilon > 0$ such that*

$$\mathbb{E}[u'(\alpha + \beta X)] < +\infty \quad \text{and} \quad \mathbb{E}\Big[u'(\alpha + \beta X)^{1+\epsilon} |X|^{2+2\epsilon}\Big] < +\infty \quad \text{for all } \alpha, \beta \in \mathbb{R}. \tag{2.1}$$

The above market structure is deliberately simple, as the paper's main focus is on preferences, which follow the reference-dependent framework of Kőszegi and Rabin (2006). The total welfare of a payoff Z results from the usual expected utility $\mathbb{E}[u(Z)]$, plus a further contribution that reflects the disappointment or satisfaction from Z in relation to some stochastic reference payoff B. Such contribution is computed

[1] The *mean-absolute difference* of a random variable X is defined as $\mathbb{E}[|X - Y|]$, where Y is independent of X and identically distributed.

[2] The case $X = 0$ almost surely (a.s.) is trivial and hence excluded here.

[3] The boundedness of $f(\cdot)$ can be replaced with the weaker, but more cumbersome condition $\int_{\mathbb{R}} f(x)^{2+\frac{1}{\epsilon}}\, dx < +\infty$, with the same $\epsilon > 0$ as in (2.1).

as follows: upon receiving the payoff Z, the agent evaluates the utility surprise $u(Z) - u(B)$ according to some gain-loss function $\nu(\cdot)$ that is kinked at the origin to reflect the higher disappointment from a given utility loss than the satisfaction from a utility gain of equal size. The resulting gain-loss $\nu(u(Z) - u(B))$ is then aggregated across all possible values of Z and B, each pair weighted according to its respective probability.

The above informal description crystallizes into the following definition.

DEFINITION 2.2 (REFERENCE-DEPENDENT UTILITY (KŐSZEGI AND RABIN, 2006)). The *gain-loss* function $\nu : \mathbb{R} \to \mathbb{R}$ is of the form[4]

$$\nu(x) := \nu_+(x^+)\, \mathbb{1}_{[0,+\infty)}(x) - \nu_-(x^-)\, \mathbb{1}_{(-\infty,0)}(x)\,, \quad \text{for all } x \in \mathbb{R},$$

where $\nu_\pm : [0, +\infty) \to \mathbb{R}$ satisfy the following assumptions:

(A1) $\nu_\pm(\cdot)$ are continuous on $[0, +\infty)$, strictly increasing on $[0, +\infty)$, twice-differentiable on $(0, +\infty)$, and $\nu_\pm(0) = 0$;

(A2) (RISK AVERSION ON GAINS AND RISK PROPENSITY ON LOSSES) $\nu''_\pm(x) \le 0$ for all $x > 0$;

(A3) (LOSS AVERSION) $\nu_+(y) - \nu_+(x) < \nu_-(y) - \nu_-(x)$ for all $x, y \in (0, +\infty)$ with $x < y$, and

$$\lambda := \frac{\nu'_+(0)}{\nu'_-(0)} \in (0, 1)\,, \tag{2.2}$$

where $\nu'_\pm(0)$ denote the right (+) and left (−) derivatives of $\nu(\cdot)$ at 0.

The *reference-dependent utility* of a payoff Z with respect to the reference B is defined as

$$\begin{aligned} U(Z|\,B) &:= \mathbb{E}[u(Z)] + \mathbb{E}\left[\int_{\mathbb{R}} \nu(u(Z) - u(b))\, d\mathbb{P}_B(b)\right] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} [u(z) + \nu(u(z) - u(b))]\, d\mathbb{P}_B(b)\, d\mathbb{P}_Z(z)\,, \end{aligned} \tag{2.3}$$

where $\mathbb{P}_Z$ and $\mathbb{P}_B$ are the probability laws of Z and B, respectively.[5]

Though Kőszegi and Rabin (2006) define reference-dependent utility for general S-shaped gain-loss functions $\nu(\cdot)$, which make agents potentially risk-seeking in losses, this paper focuses on a more parsimonious setting, in which $\nu(\cdot)$ is piecewise linear for gains and losses, with a concave kink at zero that preserves loss aversion.

ASSUMPTION 2.3. *For some* $\eta \in (0, 1)$,

$$\nu_+(x) := \frac{\lambda\eta}{1-\eta}x \quad \textit{and} \quad \nu_-(x) := \frac{\eta}{1-\eta}x, \quad \textit{for all } x \in [0, +\infty)\,.$$

[4] Here, $x^\pm := \max\{\pm x, 0\}$ for all $x \in \mathbb{R}$. In addition, $\mathbb{1}_A : X \to \{0, 1\}$ denotes the *indicator function* of the set $A \subseteq X$, defined as

$$\mathbb{1}_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

[5] Note that the product measure $\mathbb{P}_B \times \mathbb{P}_Z$ in (2.3) reflects the evaluation of the gain-loss function, in which each outcome of Z is compared to all possible values of the reference B. Thus, the above expressions could be written in the appealing form

$$U(Z|\,B) = \mathbb{E}[u(Z)] + \mathbb{E}[\nu(u(Z) - u(B))]\,,$$

where Z and B are independent random variables. This expression is more compact, but it is also partly misleading, as it suggests that the random variables are compared outcome-by-outcome. While this is technically true in the product space $(\Omega \times \Omega, \mathscr{F} \otimes \mathscr{F}, \mathbb{P} \otimes \mathbb{P})$, it is conspicuously false in the original probability space. In fact, all that is necessary to define (2.3) is the distribution of B, which may not even be supported by the probability space Ω.

The advantage of this parametrization is to reduce reference-dependence to the two parameters $\eta, \lambda \in (0,1)$. The parameter λ is a measure of loss tolerance (cf. Abdellaoui, Bleichrodt, and Paraschiv, 2007), defined as the ratio between sensitivity to gains and sensitivity to losses, as in Benartzi and Thaler (1995); Köbberling and Wakker (2005): $\lambda \downarrow 0$ represents extreme sensitivity to losses, which makes gains irrelevant, while $\lambda = 1$ recovers the usual case of a smooth utility, with equal sensitivity for gains and losses. The parameter η controls the relative weights of classical versus reference-dependent utility in the overall objective, with $\eta = 0$ recovering the classical setting, and $\eta \uparrow 1$ the pure reference-dependent limit. (Elsewhere in the literature, e.g., Tversky and Kahneman (1992), the symbol λ denotes *loss aversion*, defined as the ratio between small losses and equal-sized gains, which corresponds to $1/\lambda$ in our notation. This paper prefers to focus on loss tolerance, which is between zero and one, and hence allows to display parameter regimes in the foregoing figures without axis distortions.)

Reference-dependent preferences lead to the two central concepts of *personal equilibrium* and *preferred personal equilibrium*. Informally, a personal equilibrium is a payoff that is optimal when used as reference: if the agent takes such a strategy as a reference, then the strategy is indeed chosen. In this sense, personal equilibria are the only rational references, which are actually selected once adopted.

As personal equilibria can be numerous and each of them may have a different utility, the question arises of which personal equilibrium is optimal, leading to the concept of preferred personal equilibrium, a personal equilibrium that is not surpassed by any other. A rational forward-looking agent, unencumbered by a legacy reference, necessarily chooses a preferred personal equilibrium.

Definition 2.4 (Personal equilibria).

(i) A portfolio ϕ is a *personal equilibrium* with initial wealth w_0 if

$$\sup_{\varphi\in\mathbb{R}} U\Big(V_T^{\varphi}\Big|\, V_T^{\phi}\Big) = U\Big(V_T^{\phi}\Big|\, V_T^{\phi}\Big), \tag{2.4}$$

and $\mathrm{PE}(w_0) \subseteq \mathbb{R}$ denotes the *set of personal equilibria*.

(ii) A portfolio $\phi \in \mathrm{PE}(w_0)$ is a *preferred personal equilibrium* if

$$v^*(w_0) := \sup_{\varphi\in\mathrm{PE}(w_0)} U\Big(V_T^{\varphi}\Big|\, V_T^{\varphi}\Big) = U\Big(V_T^{\phi}\Big|\, V_T^{\phi}\Big),$$

and $\mathrm{PPE}(w_0) \subseteq \mathbb{R}$ denotes the *set of preferred personal equilibria*.

3 Main Results

3.1 Linear utility

The first result of the paper identifies personal equilibria in the special case of a linear utility function $u(\cdot)$. In the context of reference-dependence, a linear utility does not imply risk neutrality, in view of loss aversion in the reference-dependent component.

The theorem below finds that three regimes arise: *(i)* there are no personal equilibria; *(ii)* the safe portfolio is the unique personal equilibrium; and *(iii)* any risky position with positive expected return (including zero) is a personal equilibrium. These conclusions are illustrated in Figure 1, which displays the parameter combinations in which each regime arises.

Theorem 3.1. *Let $u(\cdot)$ be linear, $\mu := \mathbb{E}[X] \neq 0$, and let Assumptions 2.1 and 2.3 hold. Also, set $\mu_\pm := \mathbb{E}[X^\pm]$, and denote the* Gini index *of the distribution of X by*

$$G := \frac{1}{|\mu|}\int_{\mathbb{R}} \mathbb{P}\{X \le x\}\,\mathbb{P}\{X > x\}\,dx.$$

(i) (Personal equilibria)

(a) *If*

$$1 - \eta(1-\lambda) \leq \frac{\mu_+}{\mu_-} \leq \frac{1}{1-\eta(1-\lambda)} \tag{3.1}$$

and

$$1 - \eta(1-\lambda) \neq \frac{G-1}{G+1} \tag{3.2}$$

both hold, then $PE(w_0) = \{0\}$, *where* 0 *is the portfolio with all wealth in the safe asset.*

(b) *If* (3.1) *holds but* (3.2) *fails, then* $PE(w_0) = \{\phi \in \mathbb{R} : \phi\mu \geq 0\}$.

(c) *If* (3.1) *fails, then* $PE(w_0) = \varnothing$.

(ii) (Preferred personal equilibria)

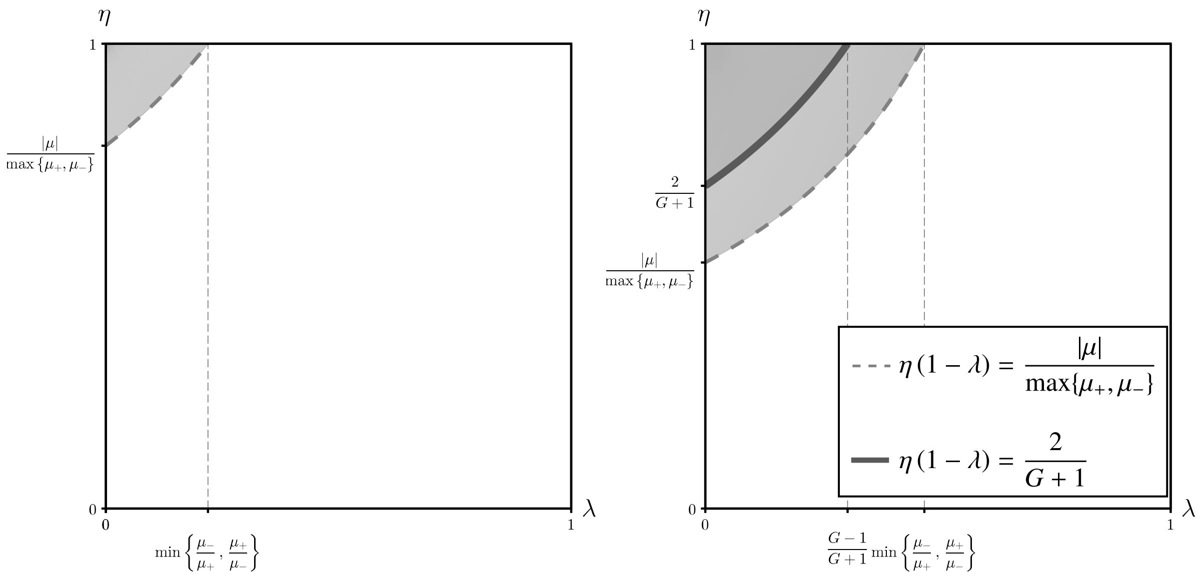
If (3.1) *holds, then* $PPE(w_0) = \{0\}$. *Otherwise,* $PPE(w_0) = \varnothing$.

Proof. See Appendix A.2. □

Remark 3.2. Condition (3.1) is equivalent to the condition

$$\frac{1+\nu_+'(0)}{1+\nu_-'(0)} \leq \frac{\mu_+}{\mu_-} \leq \frac{1+\nu_-'(0)}{1+\nu_+'(0)} \tag{3.3}$$

identified in Kőszegi and Rabin (2007, Proposition 11(i)) as necessary for a zero lottery to be a personal equilibrium, and sufficient for it to be preferred to *sufficiently small* favorable bets (with a safe reference). By contrast, Theorem 3.1 implies, under the same condition, that the safe investment combined with the safe reference is strictly better than *all* risky portfolios combined with the safe reference, hence (3.3) is also sufficient for the existence of the safe personal equilibrium. Note also that the potentially unbounded random variables considered here raise the integrability issues addressed in Lemma A.3 below.



(a) $G \leq 1$. (b) $G > 1$.

Figure 1: *The set of personal equilibria of an investor with linear utility (in a market where* $\mu \neq 0$*) is defined by the relation between the investor's loss tolerance* λ *on the x-axis and reference-dependence* η *on the y-axis. The three regions represent the sets of personal equilibria for all combinations of the parameters* λ *and* η: $PE(w_0) = \varnothing$ *on* ▭, $PE(w_0) = \{0\}$ *on* ▭, *and* $PE(w_0) = \{\phi \in \mathbb{R} : \phi\mu \geq 0\}$ *on* ━.

Even a cursory look at this result immediately highlights some stark differences from usual portfolio theory. First, an asset's appeal is measured not by its expected return, but rather by its *gain-loss ratio* μ_+/μ_-. Bernardo and Ledoit (2000) postulated such ratio as the central element of their asset pricing model: here it arises from the combination of loss-aversion with reference-dependent utility. Second, the risk of an asset is not described by variance, but rather by the Gini index G, a measure of dispersion introduced by Gini (1912) and commonly used in the inequality literature, but apparently novel in portfolio choice. Note, however, that while the Gini index of a distribution with positive support varies between 0 (no dispersion) and 1 (extreme dispersion), the Gini index of a random variable taking positive as well as negative values, such as the excess return considered here, can take values above one (Lemma A.7 below offers bounds for the Gini index in terms of the Sharpe ratio).

The main message of Theorem 3.1 is that, if expected gains are not significantly different from expected losses, as captured by the *gain-loss ratio* μ_+/μ_-, then the only risk-neutral preferred personal equilibrium is the fully safe portfolio. That the safe portfolio is a personal equilibrium is not surprising in view of loss aversion, which creates a tension between positive expected returns, emphasized by classical utility, and expected losses, emphasized by the reference-dependent component. The deeper question is whether other personal equilibria exist.

Condition (3.2) characterizes the uniqueness of the safe personal equilibrium, which holds unless the *Gini coefficient* of the risky asset has the critical value for which (3.2) fails (i.e., equality holds). Such a result appears puzzling at first, as the safe asset is the unique personal equilibrium for values of $1-\eta(1-\lambda)$ both below and above, but not equal to $(|G|-1)/(|G|+1)$. Behind such apparent singularity lie two sharply different reasons for a unique equilibrium below and above $1-\eta(1-\lambda)$. Below such threshold, loss aversion is strong enough to make an investor wish for a safer portfolio, no matter what the reference payoff: the safe personal equilibrium is "stable".

Above the threshold, the opposite occurs: loss aversion is so weak that any risky reference payoff encourages even more risk (and return), whence equilibrium fails unless the reference portfolio is safe, which represents an "unstable" equilibrium. At the threshold, loss aversion encourages neither more nor less risk taking, making any strategy with positive expected return a personal equilibrium. This interpretation is further supported by Theorem 3.4 below in the context of risk aversion, which leads to multiple equilibria. Note also that[6]

$$\frac{G-1}{G+1} = -\frac{\mathbb{E}[X \wedge Y]}{\mathbb{E}[X \vee Y]}, \tag{3.4}$$

where the random variable Y is independent of, and identically distributed to the excess return. Thus, the right-hand side of (3.4) can be interpreted as the opposite of a *min-max ratio*, a scale-invariant attribute that describes how far the average minimum of two independent outcomes is from the average maximum.[7]

In summary, reference dependence induces a delicate tradeoff between loss aversion and gain-loss ratios even for linear utilities, generating two main regimes: either the safe asset is the only preferred personal equilibrium, or no equilibrium exists. The preferred safe equilibrium region contains a borderline case with infinitely many equilibria.

Remark 3.3 (Long Positions Constraint). The non-existence of personal equilibria in Theorem 3.1 stems from the absence of constraints on risky positions, which lead a risk-neutral utility maximizer to take arbitrarily large risks when they are sufficiently attractive. (Put differently, neither the set of trading strategies nor the superlevel sets of the objective are compact—unlike Theorem 3.4 below.[8]) Alternative assumptions of interest include the possibility that neither leverage nor short sales are allowed, whence the strategy must lie in the interval $[0, w_0]$, and a personal equilibrium always exists. An inspection of the proof in the appendix reveals that, under such constraint, the statement of Theorem 3.1

6 Here, $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$, for all $x, y \in \mathbb{R}$.

7 Note also that the min-max ratio is strictly between -1 and 1, and that the Gini index and min-max ratio of a (purely atomic) Dirac law δ_{x_0}, for some $x_0 \in \mathbb{R} \setminus \{0\}$, are equal to 0 and 1, respectively.

8 In this respect, our setting differs from the standard environment of Kőszegi and Rabin (2006), in which the action set is compact.

changes as follows:

(i') (PERSONAL EQUILIBRIA)

(a') If, in addition to (3.1),

$$1 - \eta(1-\lambda) > \frac{G-1}{G+1}, \tag{3.5}$$

then $\mathrm{PE}(w_0) = \{0, w_0\}$.

(b') If, in addition to (3.1),

$$1 - \eta(1-\lambda) < \frac{G-1}{G+1},$$

then $\mathrm{PE}(w_0) = \{0\}$.

(c') If (3.1) holds but (3.2) fails, then $\mathrm{PE}(w_0) = [0, w_0]$.

(d') If (3.1) fails, then $\mathrm{PE}(w_0) = \{w_0\}$.

(ii') (PREFERRED PERSONAL EQUILIBRIA)

If (3.1) holds, then $\mathrm{PPE}(w_0) = \{0\}$. Otherwise, $\mathrm{PPE}(w_0) = \{w_0\}$.

3.2 Concave utility

With concave utilities, risk aversion arises from both the utility function and the loss aversion in the reference-dependent component. The next result characterizes the optimal solution in this setting, in which a personal equilibrium always exists. In fact, two personal equilibria typically compete for optimality: one is the safe asset, as for linear utility. The other one involves a mix of safe and risky investments, and converges to the usual maximizer of expected utility as reference-dependence vanishes.

THEOREM 3.4. *Let $u'(\cdot)$ be strictly decreasing with $u'(-\infty) = +\infty$, $\mu \neq 0$, and let Assumptions 2.1 and 2.3 hold.*

(i) (PERSONAL EQUILIBRIA)

(a) *If*

$$1 - \eta(1-\lambda) \leq \frac{\mu_+}{\mu_-} \leq \frac{1}{1-\eta(1-\lambda)} \tag{3.1}$$

and

$$1 - \eta(1-\lambda) > \frac{G-1}{G+1} \tag{3.5}$$

both hold, then $PE(w_0) = \{0, \theta^*\}$ for some $\theta^* \equiv \theta^*(\eta, \lambda)$ such that $\theta^* \mu > 0$.

(b) *If (3.1) holds but (3.5) fails, then $PE(w_0) = \{0\}$.*

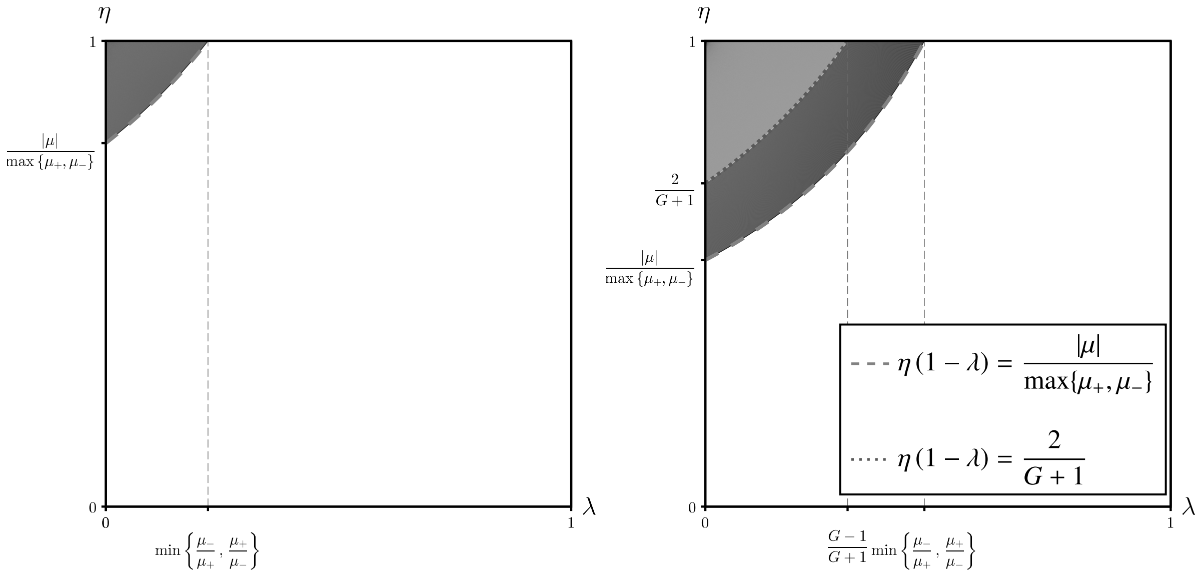
(c) *If (3.1) fails, then $PE(w_0) = \{\theta^*\}$.*

(ii) (PREFERRED PERSONAL EQUILIBRIA)

If (3.1) holds and $G \geq 1$, then $PPE(w_0) = \{0\}$. If (3.1) fails, then $PPE(w_0) = \{\theta^*\}$.

Proof. See Appendix A.2. □

REMARK 3.5. Theorem 3.4 identifies the set of personal equilibria in all cases, and the preferred personal equilibrium in all cases other than $G < 1$ combined with (3.1). In such a case, determining which personal equilibrium is preferred requires the comparison of the values of the respective utilities, on a model-by-model basis. We did not find model-free conditions that identify the preferred personal equilibrium in this case.



(a) $G \le 1$.

(b) $G > 1$.

Figure 2: *Regions of different sets of personal equilibria in the parameter space* (λ, η) *for an investor with strictly concave utility and unbounded marginal (in a market where* $\mu \neq 0$*):* $PE(w_0) = \{\theta^*\}$ *on* , $PE(w_0) = \{0, \theta^*\}$ *on* , *and* $PE(w_0) = \{0\}$ *on* . *The dashed curve and the dotted curve represent the existence boundary of the safe personal equilibrium and of the risky personal equilibrium, respectively.*

Figure 2 displays the regimes arising for different combinations of η and λ. Weak loss aversion (λ near one) or reference-dependence (η near zero) generate a single personal equilibrium that includes both safe and risky investments, as in the usual expected-utility setting. In this regime, the asymmetry between gains and losses is not sufficient to alter the qualitative structure of the solution.

As reference dependence and loss aversion increase, the safe equilibrium emerges, coexisting with the classical one (darker region in Figure 2). In such a regime, either equilibrium can be preferred, depending on their respective values, which in turn depend on the specific utility function. If the return of the risky asset is relatively concentrated ($G \le 1$), no other regime is possible. However, if dispersion is high ($G > 1$), then a third regime arises, in which the safe investment survives as the only, hence preferred, personal equilibrium.

REMARK 3.6 (LONG POSITIONS CONSTRAINT). If leverage and short sales are excluded, the conclusions of Theorem 3.4 remain valid if the initial capital is sufficiently large and $\mathbb{P}\{X < -1 - r\} > 0$. By contrast, the borrowing constraint is binding whenever the initial capital is small, whence $\mathrm{PE}(w_0) = \mathrm{PPE}(w_0) = \{0\}$ if (3.1) holds, and $\mathrm{PE}(w_0) = \mathrm{PPE}(w_0) = \varnothing$ otherwise.

The last theorem of this subsection describes the sensitivity of the only risky equilibrium (when it exists) in Theorem 3.4 to model parameters, such as gain-loss sensitivity, loss tolerance, initial capital, and risk aversion.

THEOREM 3.7. *Let* $u'(\cdot)$ *be strictly decreasing with* $u'(-\infty) = +\infty$, $\mu \neq 0$, *and let Assumptions 2.1 and 2.3 as well as condition* (3.5) *hold. Also, denote by* θ *the unique solution of the classical utility maximization problem, i.e.,*

$$\sup_{\varphi \in \mathbb{R}} \mathbb{E}\left[u\left(V_T^{\varphi}\right)\right] = \mathbb{E}\left[u\left(V_T^{\theta}\right)\right], \tag{3.6}$$

and by $\theta^*(\eta, \lambda)$ *the unique nonzero (i.e., risky) personal equilibrium (which exists by* (3.5)*).*

(i) (SENSITIVITY TO PREFERENCE PARAMETERS)

(a) (CONTINUITY) *The mapping* $(\eta,\lambda)\mapsto\theta^*(\eta,\lambda)$ *is continuous.*

(b) (REFERENCE DEPENDENCE INDUCES LESS RISK TAKING) *For all* η,λ *satisfying* (3.5),

$$\left|\theta^*(\eta,\lambda)\right| < |\theta|\,. \tag{3.7}$$

(c) (MONOTONICITY) *The mapping* $\lambda\mapsto\theta^*(\eta,\lambda)$ *is strictly increasing for any* $\eta\in(0,1)$, *and* $\eta\mapsto\theta^*(\eta,\lambda)$ *is strictly decreasing for any* $\lambda\in(0,1)$.

(d) (CLASSICAL UTILITY LIMITS) *For any* $\eta,\lambda\in(0,1)$, $\lim_{\lambda\to1}\theta^*(\eta,\lambda)=\theta$ *and* $\lim_{\eta\to0}\theta^*(\eta,\lambda)=\theta$.

(e) (PURE REFERENCE-DEPENDENT LIMITS) *If* $G>1$, *then for all* $\bar\eta,\bar\lambda\in(0,1)$ *for which* (3.2) *fails,*

$$\lim_{(\eta,\lambda)\to(\bar\eta,\bar\lambda)}\theta^*(\eta,\lambda)=0.$$

If $G=1$, *then*

$$\lim_{(\eta,\lambda)\to(1,0)}\theta^*(\eta,\lambda)=0.$$

If $G<1$, *then there exists* $\bar\theta$ *with* $\bar\theta\mu>0$ *and* $|\bar\theta|<|\theta^*|$ *such that*

$$\lim_{(\eta,\lambda)\to(1,0)}\theta^*(\eta,\lambda)=\bar\theta.$$

(ii) (SENSITIVITY TO RISK AVERSION)

Let $\eta,\lambda\in(0,1)$ *satisfy* (3.5), *and let* $u_i(\cdot)$, $i\in\{1,2\}$, *be two utility functions such that* $u_i'(\cdot)$ *is strictly decreasing with* $u_i'(-\infty)=+\infty$. *If* $u_2(\cdot)$ *is a* concave monotone transformation *of* $u_1(\cdot)$, *i.e., there exists a strictly increasing, concave and differentiable function* $\rho(\cdot)$ *such that*

$$u_2(x)=\rho(u_1(x))\quad \text{for all } x\in\mathbb{R}, \tag{3.8}$$

then $\left|\theta_1^*\right|\geq\left|\theta_2^*\right|$, *where* θ_i^* *denotes the unique risky personal equilibrium for the reference-dependent problem* (2.4) *with utility* $u_i(\cdot)$.

(iii) (SENSITIVITY TO INITIAL CAPITAL)

Let $\eta,\lambda\in(0,1)$ *satisfy* (3.5), *and assume further that* $u(\cdot)$ *is twice-differentiable. Also, denote the* Arrow-Pratt coefficient of absolute risk aversion *(Arrow, 1965; Pratt, 1964) of* $u(\cdot)$ *by*

$$\mathrm{ARA}_u(x):=-\frac{u''(x)}{u'(x)},\quad \text{for all } x\in\mathbb{R}.$$

If $\mathrm{ARA}_u(\cdot)$ *is non-increasing (respectively, constant or non-decreasing), then* $\partial\theta^*/\partial w_0$ *is non-negative (respectively, zero or non-positive).*

Proof. See Appendix A.2. □

Property *(i)(a)* states the continuity of the personal equilibrium with respect to the preference parameters, meaning that slight changes in preference parameters produce only slight changes in the risky personal equilibrium. Equation (3.7) shows that the personal equilibrium implies a lower investment in the risky asset, in view of loss aversion. The interpretation of *(i)(c)* is that more loss tolerance leads to a riskier position; likewise, the larger the gain-loss sensitivity, the smaller the risky position. Property *(i)(d)* stipulates that, as the gain-loss sensitivity vanishes either through η or λ, the personal equilibrium boils down to the unique utility-maximizing portfolio.

Part *(i)(e)* describes the impact of the return dispersion G on the optimal portfolio. A high dispersion ($G\geq1$) has a predictable effect: the personal equilibrium degenerates to the safe investment as the reference-dependent component overwhelms expected utility (i.e., η near 1 and λ near 0 when $G=1$; if

$G > 1$, however, then it is not possible for the pair (η, λ) to even approach $(0, 1)$, since it can only reach as high as the threshold for existence of the risky personal equilibrium).

A more surprising phenomenon occurs for more concentrated distributions ($G < 1$): then the unique personal equilibrium converges to a nontrivial, nonzero limit, even for an agent solely focused on loss aversion, and insensitive to gains ($\eta \uparrow 1$, $\lambda \downarrow 0$). At first glance, such a result is puzzling, as the investor has nothing to gain from risk taking. The key here is that an ambitious reference payoff can induce risky investments, even if gains are disregarded, purely to keep up with the reference by avoiding losses. Put differently, for an investor with a high reference payoff and a moderately concentrated return, opting for the safe asset is an unpalatable choice, as it entails larger losses from the reference than a risky investment that on average comes closer to such target. Thus, such an investor is trapped by a high reference in a non-preferred personal equilibrium, which is inferior to a safe investment *combined* with the safe reference.

Turning to *(ii)*, first recall that, if the utilities $u_1(\cdot)$ and $u_2(\cdot)$ are both twice-differentiable then, by Arrow-Pratt's theorem (Arrow, 1965; Pratt, 1964), condition (3.8) is equivalent to replacing $u_1(\cdot)$ with some $u_2(\cdot)$ with higher risk aversion than $u_1(\cdot)$. Hence, as for expected utility, higher risk aversion leads to safer investments.

Finally, property *(iii)* recovers the same wealth effects for reference-dependent utility as for classical utility: if an agent has constant absolute risk aversion, then the risky personal equilibrium is independent of the initial capital; if an agent's absolute risk aversion is decreasing (respectively, increasing) in wealth, then the stock is a normal good (respectively, an inferior good)—i.e., the optimal amount allocated to the risky asset increases (respectively, decreases) with the initial capital.

Remark 3.8 (Recovering Risk Neutrality). It is tempting to view Theorem 3.1 as the limit case of Theorem 3.4 as risk aversion vanishes to risk neutrality, but there are perils in such an interpretation. As intuition suggests, in the model in the next section the risky personal equilibrium becomes arbitrarily large as the utility function becomes linear (Proposition 4.2*(iv)*)—the risky personal equilibrium disappears. Yet, contrary to intuition, it is possible to construct sequences of concave utilities that converge pointwise to a linear function, while the corresponding risky personal equilibria remain confined in a bounded interval (Lemma A.6 below). In summary, while risk aversion can be zero only in one way, it can vanish in many ways, and personal equilibria may diverge, converge, or oscillate.

3.3 Ramifications and extensions.

3.3.1 More general gain-loss functions.

Piecewise-linear gain-loss functions conveniently confine the effect of reference-dependence to the loss aversion at the reference point. The question is to what extent the results in the previous section carry over to more general gain-loss functions.

Throughout this subsection, assume that $u(\cdot)$ is twice-differentiable, and suppose that Assumption 2.3 is replaced by the weaker:

Assumption 3.9. *For all $x, y \in \mathbb{R}$ such that $x < y$,*

$$-u''(x)\left(1 + \nu_-'(u(y) - u(x))\right) \geq -u'(x)^2\, \nu_-''(u(y) - u(x))\,. \tag{3.9}$$

This condition ensures that, while investors may be risk-seeking on losses, the concavity of their utility $u(\cdot)$ more than compensates the convexity of $\nu_-(\cdot)$, so that the overall reference-dependent problem (2.4) remains globally concave. Note that (3.9) holds in particular whenever $\nu_-(\cdot)$ is a linear function.

With this more general gain-loss function, (3.3) remains necessary and sufficient for the safe portfolio to be a personal equilibrium. By contrast, some differences arise for risky personal equilibria, which are addressed in the following result.

LEMMA 3.10. *Let $\mu \neq 0$, and let Assumptions 2.1 and 3.9 hold.*

(i) (LINEAR UTILITY)

Let $u(\cdot)$ be linear. Also, set $\nu'_{\pm}(+\infty) := \lim_{x\to+\infty} \nu'_{\pm}(x) \in [0,+\infty)$ and

$$\lambda_\infty := \frac{\nu'_+(+\infty)}{\nu'_-(+\infty)}.$$

(a) If

$$\frac{G-1}{G+1} \in \left(1 - \frac{\nu_-(0)\,(1-\lambda_\infty)}{1+\nu_-(0)}, 1 - \frac{\nu_-(0)\,(1-\lambda)}{1+\nu_-(0)}\right), \tag{3.10}$$

then $PE(w_0) \setminus \{0\}$ is non-empty and bounded, and all risky personal equilibria have the same sign as μ.

(b) If

$$\frac{G-1}{G+1} \notin \left[1 - \frac{\nu_-(0)\,(1-\lambda_\infty)}{1+\nu_-(0)}, 1 - \frac{\nu_-(0)\,(1-\lambda)}{1+\nu_-(0)}\right], \tag{3.11}$$

then $PE(w_0) \setminus \{0\} = \emptyset$.

(ii) (CONCAVE UTILITY)

Let $u'(\cdot)$ be strictly decreasing with $u'(-\infty) = +\infty$. If

$$1 - \frac{\nu'_-(0)\,(1-\lambda)}{1+\nu'_-(0)} > \frac{G-1}{G+1}, \tag{3.12}$$

then $PE(w_0) \setminus \{0\}$ is non-empty and bounded, and all risky personal equilibria have the same sign as μ.

Proof. See Appendix A.2. □

The main difference of this result from the piecewise-linear case is that the number of risky personal equilibria is not determined a priori. For an investor with linear utility, first note that $u(\cdot)$ can override risk-propensity on losses only if $\nu_-(\cdot)$ is linear. As conditions (3.10) and (3.11) show, the existence of risky personal equilibria is now also dependent on the new parameter λ_∞; as in He and Zhou (2011), we denote it as the *large-loss tolerance* (i.e., the analogue of the loss tolerance parameter λ, but for large rather than small payoffs). While part *(i)(a)* states that there is at least one risky personal equilibrium if (3.10) holds, part *(i)(b)* is a partial converse result, which makes condition (3.11) essentially sharp, as it leaves open only the borderline case

$$\frac{G-1}{G+1} \in \left\{1 - \frac{\nu_-(0)\,(1-\lambda_\infty)}{1+\nu_-(0)}, 1 - \frac{\nu_-(0)\,(1-\lambda)}{1+\nu_-(0)}\right\}$$

which needs to be examined model by model, with the exception of $\nu_+(\cdot)$ linear (which falls in the case *(i)(b)* of Theorem 3.1).

Turning to a strictly concave utility with unbounded marginal, it is still possible to establish the existence of a risky personal equilibrium with positive expected return under condition (3.12), which is a generalization of (3.5). While uniqueness is no longer guaranteed, for all risky personal equilibria the amount of wealth invested in the stock never exceeds a certain level.

3.3.2 Mixed strategies.

The paper defines a portfolio as a *pure strategy*, a fixed number of shares that the investor chooses at the beginning of the period, consistently with usual portfolio theory. Indeed, a *mixed strategy*—a randomized number of shares—would not serve a classic utility maximizer well, as the added noise from randomization would increase the overall portfolio risk. Yet, in the present setting the question arises of whether randomization can improve an investor's prospects by increasing the reference-dependent contribution. It turns out that it cannot.

To define mixed strategies, consider the product probability space $(\Omega \times \Omega', \mathscr{F} \otimes \mathscr{F}', \mathbb{P} \otimes \mathbb{P}')$, where $(\Omega, \mathscr{F}, \mathbb{P})$ is the original probability space describing the uncertainty in the market, while the extrinsic randomness in the investors' strategies is modeled by some random variable ζ defined on the second probability space $(\Omega', \mathscr{F}', \mathbb{P}')$. Let $\mathscr{F}'_0$ be the sigma-algebra generated by ζ. A *mixed strategy* starting from initial capital w_0 is any $\mathscr{F}'_0$-measurable, integrable random variable ψ with terminal value equal to

$$V_T^{\psi} = \widetilde{w_0} + \psi X,$$

and denote the set of such mixed (or randomized) strategies by $\mathscr{R}$. The reference-dependent utility is defined by (2.3), the difference being that the integrals are now on the product space $\Omega \times \Omega'$ with respect to the product measure $\mathbb{P} \otimes \mathbb{P}'$.

Intuitively, it is not worth enlarging the set of strategies to include mixed ones, as these only add more noise to the investors' payoff without improving welfare; this intuition is confirmed by the following lemma. Roughly speaking, part *(i)* states that an agent planning to adopt a mixed strategy is actually strictly better off choosing the associated mean (pure) strategy, thus precluding mixed-strategy personal equilibria; by part *(ii)*, for an investor with a pure-strategy reference, taking a mixed strategy never leads to more satisfaction, hence the set of (pure) personal equilibria is not affected by this relaxation of the set of strategies.

Lemma 3.11. *Let Assumptions 2.1 and 3.9 hold, and assume that either $u(\cdot)$ is linear or $u'(\cdot)$ is strictly decreasing with $u'(-\infty) = +\infty$. Then:*

(i) *For all $\psi \in \mathscr{R} \setminus \mathbb{R}$,*

$$U\Big(V_T^{\psi}\Big|V_T^{\psi}\Big) < U\Big(V_T^{\bar{\psi}}\Big|V_T^{\psi}\Big),$$

where $\bar{\psi} := \mathbb{E}[\psi] \in \mathbb{R}$.

(ii) *For all $\phi \in \mathbb{R}$,*

$$\sup_{\varphi \in \mathbb{R}} U\Big(V_T^{\varphi}\Big|V_T^{\phi}\Big) = \sup_{\psi \in \mathscr{R}} U\Big(V_T^{\psi}\Big|V_T^{\phi}\Big). \tag{3.13}$$

Proof. See Appendix A.2. □

4 Example

This section examines in detail a concrete model, focusing on exponential utility combined with normally distributed returns.

Assumption 4.1. *The stock's excess return X has normal distribution with mean $\mu \in \mathbb{R} \setminus \{0\}$ and standard deviation $\sigma > 0$. Investors have exponential utility with constant absolute risk aversion coefficient $\gamma > 0$,*

$$u(x) := \frac{1 - e^{-\gamma x}}{\gamma}, \quad \textit{for all } x \in \mathbb{R}.$$

In the absence of reference dependence, in this setting the optimal portfolio prescribes an amount invested in the risky asset given by the usual Merton formula:

$$\theta_M := \frac{\mu}{\gamma\sigma^2}.$$

The following result shows how reference-dependent preferences affect the optimal portfolio, and characterizes the parameter restrictions under which one or two personal equilibria arise.

PROPOSITION 4.2. *Let Assumptions 2.3 and 4.1 hold. Denote by* $S := \frac{\mu}{\sigma}$ *the stock's* Sharpe ratio*, and by* $\Phi(\cdot)$ *the standard normal distribution function.*

(i) (a) (PERSONAL EQUILIBRIA)
If

$$1-\eta(1-\lambda) \leq \frac{1+S\sqrt{2\pi}e^{\frac{S^2}{2}}\Phi(S)}{1-S\sqrt{2\pi}e^{\frac{S^2}{2}}\Phi(S)} \leq \frac{1}{1-\eta(1-\lambda)} \tag{4.1}$$

and

$$1-\eta(1-\lambda) > \frac{1-\sqrt{\pi}|S|}{1+\sqrt{\pi}|S|} \tag{4.2}$$

both hold, then $PE(w_0) = \{0, \theta^*\}$*, where* θ^* *is the unique solution of the transcendental equation in* z*:*

$$\left(\mu-\gamma\sigma^2 z\right)\left(1-\eta(1-\lambda)+\eta(1-\lambda)\,\Phi\!\left(\frac{\gamma\sigma|z|}{\sqrt{2}}\right)\right) = \eta(1-\lambda)\,e^{-\left(\frac{\gamma\sigma z}{2}\right)^2}\frac{\operatorname{sgn}(z)\,\sigma}{2\sqrt{\pi}}.$$

If (4.1) *holds but* (4.2) *fails, then* $PE(w_0) = \{0\}$*. If* (4.1) *fails, then* $PE(w_0) = \{\theta^*\}$*.*

(b) (PREFERRED PERSONAL EQUILIBRIA)
If (4.2) *holds and* $|S| \leq \pi^{-\frac{1}{2}}$*, then* $PPE(w_0) = \{0\}$*. If* (4.2) *fails, then* $PPE(w_0) = \{\theta^*\}$*.*

(ii) θ^* *is strictly increasing in* λ*,* μ*; strictly decreasing in* η*; constant in* w_0*. Moreover,* $\lim_{\eta\to 0}\theta^* = \theta_M$ *and* $\lim_{\lambda\to 1}\theta^* = \theta_M$*.*

(iii) *If* $|S| < \pi^{-\frac{1}{2}}$*, then* $\lim_{(\eta,\lambda)\to(\bar\eta,\bar\lambda)}\theta^*(\eta,\lambda) = 0$ *for all* $\bar\eta, \bar\lambda$ *such that*

$$1-\bar\eta\left(1-\bar\lambda\right) = \frac{1-\sqrt{\pi}|S|}{1+\sqrt{\pi}|S|}.$$

If $|S| = \pi^{-\frac{1}{2}}$*, then* $\lim_{(\eta,\lambda)\to(1,0)}\theta^*(\eta,\lambda) = 0$*. If* $|S| > \pi^{-\frac{1}{2}}$*, then* $\lim_{(\eta,\lambda)\to(1,0)}\theta^*(\eta,\lambda) = \bar\theta$*, where* $\bar\theta$ *is the unique solution of the transcendental equation in* z*:*

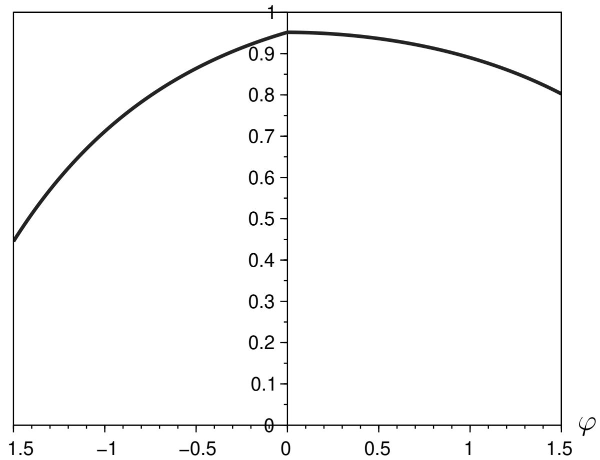
$$\left(\mu-\gamma\sigma^2 z\right)\Phi\!\left(\frac{\gamma\sigma z}{\sqrt{2}}\right) = \frac{\sigma}{2\sqrt{\pi}}e^{-\left(\frac{\gamma\sigma z}{2}\right)^2}.$$

(iv) $|\theta^*|$ *is strictly decreasing in* σ*,* γ*; and* $|\theta^*| < |\theta_M|$*. Moreover,* $\lim_{\gamma\to 0}|\theta^*| = +\infty$ *and* $\lim_{\gamma\to+\infty}\theta^* = 0$*.*

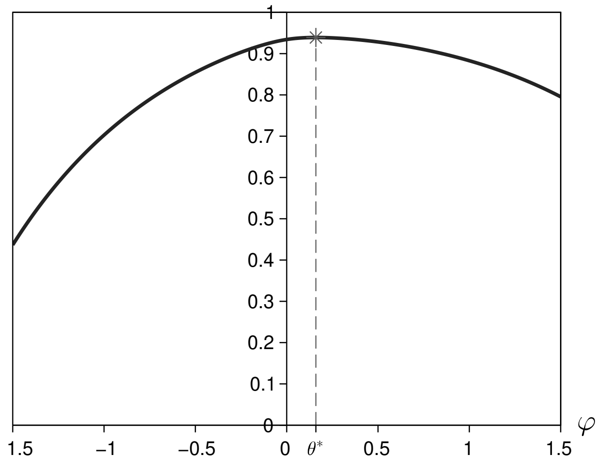
Proof. See Appendix A.2. □

The Gaussian distribution has the particularity that its Gini index is inversely proportional to the Sharpe ratio. Consequently, we can write the gain-loss ratio and the opposite of the min-max ratio as, respectively, increasing and decreasing functions of the Sharpe ratio (see (4.1) and (4.2)). As highlighted in the previous section, both of these competing analogues of the Sharpe ratio have significance, in that they give rise to different personal equilibria.

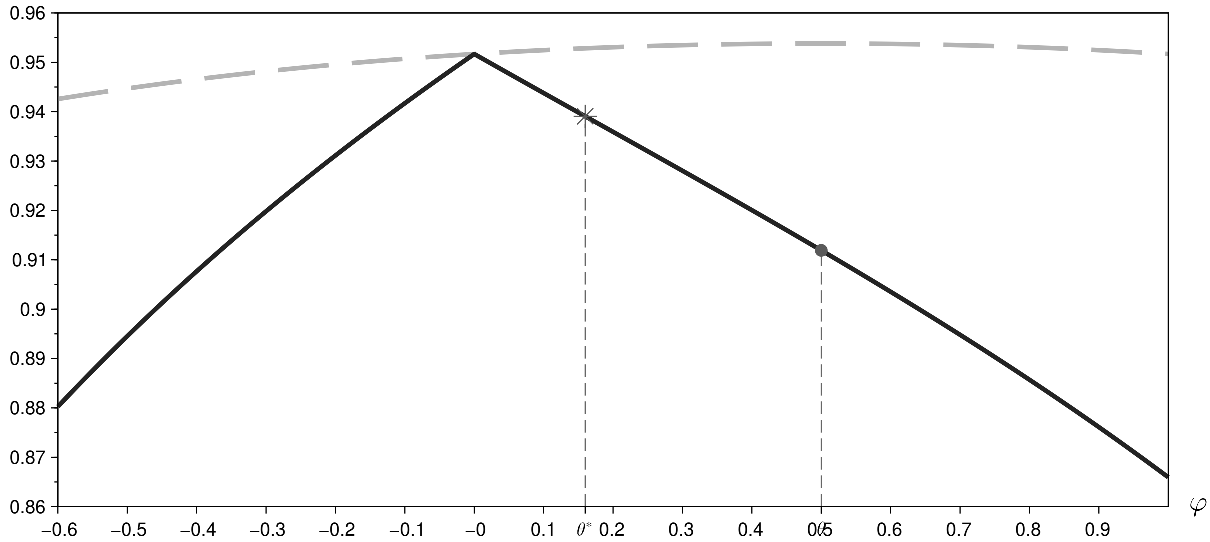
For a suitable choice of market and preference parameters, Figure 3 shows the graphs of some of the functions associated with the reference-dependent optimization problem. The risk-free rate is fixed at $r = 1\%$; the market risk premium and volatility are set equal to 6% and 20% per year, respectively. We select the loss tolerance parameter $\lambda = 0.40$ to be consistent with the value estimated by Tversky and Kahneman (1991, 1992), and the gain-loss sensitivity parameter $\eta = 0.90$ so that (4.2) holds and two personal equilibria arise. Finally, the initial capital w_0 is normalized to 1, while absolute risk aversion is set to $\gamma = 3$ so as to obtain a utility-maximizing portfolio θ_M with exposure of 50% to the risky asset, in the absence of reference dependence. Note in particular that the mapping $\varphi \mapsto U\left(V_T^\varphi \middle| V_T^\varphi\right)$, determining the set of preferred personal equilibria, is neither concave nor convex on the positive half-line. Finally, the comparative statics given by Proposition 4.2 are clearly visible in Figure 4, where the risky personal equilibrium is plotted for all possible combinations of the preference parameters.



(a) $\varphi \mapsto U\left(V_T^{\varphi} \middle| V_T^0\right)$.

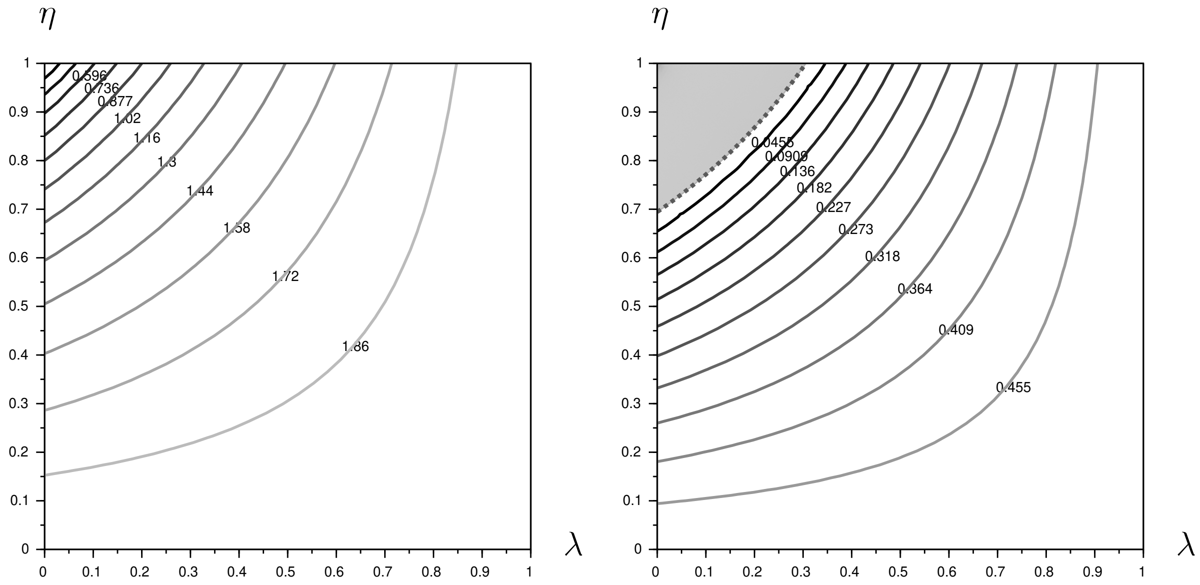


(b) $\varphi \mapsto U\left(V_T^{\varphi} \middle| V_T^{\theta^*}\right)$.



(c) $\varphi \mapsto U\left(V_T^{\varphi} \middle| V_T^{\varphi}\right)$ *(solid line) and* $\varphi \mapsto \mathbb{E}\left[u\left(V_T^{\varphi}\right)\right]$ *(dashed line).*

Figure 3: *Plots for an investor with exponential utility when the stock has normal excess return. Market and preference parameters are:* $w_0 = 1$*,* $r = 1\%$*,* $\gamma = 3$*,* $\mu = 6\%$*,* $\sigma = 20\%$*,* $\lambda = 0.40$*, and* $\eta = 0.90$*; thus,* $G \approx 1.88$*,* $\theta_M = 0.50$ *(circle marker), and* $\theta^* \approx 0.16$ *(asterisk marker).*



(a) *Here,* $\sigma = 10\%$*;*
thus, $G \approx 0.94$*,* $\theta_M = 2$*, and* $\bar{\theta} \approx 0.17$.

(b) *Here,* $\sigma = 20\%$*;*
thus, $G \approx 1.88$ *and* $\theta_M = 0.50$.

Figure 4: *Level curves of the mapping* $(\eta, \lambda) \mapsto \theta^*(\eta, \lambda)$ *for an investor with exponential utility when the stock has normal excess return. Market and preference parameters are:* $w_0 = 1$*,* $r = 1\%$*,* $\gamma = 3$*, and* $\mu = 6\%$.

5 Conclusion

This paper solves the one-period portfolio selection problem under Kőszegi and Rabin reference-dependent preferences. Any risk-averse investor with sufficiently high reference-dependence and loss aversion has two competing personal equilibria—strategies that are optimal when taken as references—as individuals with identical preferences may make different investments depending on their expectations. Investors planning to participate in the stock market optimally choose to do so, while those planning to refrain from risky investments optimally hold the safe asset only. The model offers an explanation for limited stock market participation as a result of heterogeneity in references, even in the absence of participation costs.

Although reference-dependent preferences already pose substantial challenges in the one-period setting considered here, some of them stem from market incompleteness, which restricts the choice of both references and payoffs. Models of complete markets in both discrete and continuous time offer another setting with significant potential for tractability.

A Appendix

This appendix contains the proofs of the results stated in the main body of the paper. In what follows, Y is a random variable independent of, and identically distributed to X under $\mathbb{P}$.

A.1 Auxiliary results

The first lemma is a purely technical one establishing the differentiability of a given function, which is required by Lemma A.2.

Lemma A.1. *Let Assumption 2.1 hold. Let $b \in \mathbb{R} \setminus \{0\}$, and $g : \mathbb{R}^2 \to [0, +\infty)$ be a continuous function, differentiable on $D := \{(x, y) \in \mathbb{R}^2 : y - x > 0\}$. If the following conditions hold:*

(i) $\mathbb{E}[g(bX, zY)] < +\infty$ *for all* $z \in \mathbb{R}$*;*

(ii) $g(x, y) = 0$ *for all* $(x, y) \in \mathbb{R}^2 \setminus D$*;*

(iii) $x \mapsto g(x, y)$ *is non-increasing for all* $y \in \mathbb{R}$*;*

(iv) *for all* $z \in \mathbb{R}$*, there exists* $\psi_1^z : \mathbb{R} \to [0, +\infty)$ *such that* $\mathbb{E}\big[\psi_1^z(Y)\,|Y|\big] < +\infty$ *and*

$$\left|\frac{g(bx, (z+h)\,y) - g(bx, zy)}{h}\right| \mathbb{1}_{\{(z+h)y - bx > 0\}} \le \psi_1^z(y)\,|y| \quad \text{for all } (x, y) \in \mathbb{R}^2 \text{ and } 0 < |h| < 1; \quad \text{(A.1)}$$

(v) *for all* $z \in \mathbb{R}$*, there exists* $\psi_2^z : \mathbb{R} \to [0, +\infty)$ *such that* $\mathbb{E}\big[\psi_2^z(Y)^{1+\epsilon}\,|Y|^{2+2\epsilon}\big] < +\infty$ *and*

$$g(zy - |y|, zy) \le \psi_2^z(y)\,|y| \quad \text{for all } y \in \mathbb{R};$$

then the function $\Gamma : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\Gamma(z) \equiv \Gamma(b; z) := \mathbb{E}\big[g(bX, zY)\,\mathbb{1}_{\{zY - bX > 0\}}\big], \quad \text{for all } z \in \mathbb{R},$$

is differentiable on $\mathbb{R} \setminus \{0\}$*, with*

$$\Gamma'(z) = \mathbb{E}\left[\frac{\partial g}{\partial y}(bX, zY)\,Y\,\mathbb{1}_{\{zY - bX > 0\}}\right], \quad \text{for all } z \in \mathbb{R} \setminus \{0\}\,.$$

Proof. Let $b < 0$ be given (the proof in the case $b > 0$ is analogous). Note that $\Gamma(\cdot)$ is well-defined by virtue of *(i)*. Fix an arbitrary $z \in \mathbb{R} \setminus \{0\}$, and consider a sequence $\{h_n\}_{n \in \mathbb{N}}$ of nonzero real numbers converging to 0. Assume further, without loss of generality, that $|h_n| < 1$ for all $n \in \mathbb{N}$. The difference quotient of $\Gamma(\cdot)$ at z with increment h_n is equal to

$$\frac{\Gamma(z + h_n) - \Gamma(z)}{h_n} = \mathbb{E}\left[\frac{g(bX, (z + h_n)\,Y) - g(bX, zY)}{h_n}\,\mathbb{1}_{\{(z+h_n)Y - bX > 0\}}\right] + \mathbb{E}\left[g(bX, zY)\,\frac{\mathbb{1}_{\{(z+h_n)Y - bX > 0\}} - \mathbb{1}_{\{zY - bX > 0\}}}{h_n}\right]$$

for all $n \in \mathbb{N}$. We carry out the rest of the proof in two steps.

(i) We show that $\lim_{n \to +\infty} \mathbb{E}[Z_n] = \mathbb{E}\Big[\frac{\partial g}{\partial y}(bX, zY)\,Y\,\mathbb{1}_{\{zY - bX > 0\}}\Big]$, where

$$Z_n := \frac{g(bX, (z + h_n)\,Y) - g(bX, zY)}{h_n}\,\mathbb{1}_{\{(z+h_n)Y - bX > 0\}} \quad \text{for all } n \in \mathbb{N}.$$

Let ω outside of the null event $\Omega_1 := \{|X| = +\infty\} \cup \{|Y| = +\infty\} \cup \{zY = bX\} \cup \{Y = 0\}$, and let $\varepsilon'(\omega) > 0$. If $zY(\omega) - bX(\omega) < 0$, then $(z + h_n)\,Y(\omega) - bX(\omega) < 0$ for all n sufficiently large, so $\lim_{n \to +\infty} Z_n(\omega) = 0 = \frac{\partial g}{\partial y}(bX(\omega)\,, zY(\omega))\,Y(\omega)\,\mathbb{1}_{\{zY - bX > 0\}}(\omega)$. If, on the other hand, $(bX(\omega)\,, zY(\omega)) \in D$, there exists $\delta > 0$ such that

$$\left|\frac{g(bX(\omega)\,, zY(\omega) + h) - g(bX(\omega)\,, zY(\omega))}{h} - \frac{\partial g}{\partial y}\,(bX(\omega)\,, zY(\omega))\right| < \varepsilon'(\omega)$$

for all $h \in \mathbb{R}$ with $0 < |h| < \delta$. Therefore, relabeling $\varepsilon'(\omega)$ as $\varepsilon / Y(\omega)$,

$$\begin{aligned}&\left|Z_n(\omega) - \frac{\partial g}{\partial y}(bX(\omega)\,, zY(\omega))\,Y(\omega)\,\mathbb{1}_{\{zY - bX > 0\}}(\omega)\right| \\ &\quad = \left|\frac{g(bX(\omega)\,, (z + h_n)\,Y(\omega)) - g(bX(\omega)\,, zY(\omega))}{h_n Y(\omega)} - \frac{\partial g}{\partial y}(bX(\omega)\,, zY(\omega))\right| |Y(\omega)| < \varepsilon,\end{aligned}$$

since $|h_n| < \frac{\delta}{|Y(\omega)|}$ and $(z + h_n)\,Y(\omega) - bX(\omega) > 0$ for all n large enough. Furthermore, it is an immediate consequence of (A.1) that the sequence $\{Z_n\}_{n \in \mathbb{N}}$ is dominated a.s. by the integrable random variable $\psi_1^z(Y)\,|Y|$. Hence, the use of Lebesgue's dominated convergence theorem is justified.

(ii) We claim that

$$\lim_{n\to+\infty} \mathbb{E}\left[g(bX, zY)\, \frac{\mathbb{1}_{\{(z+h_n)Y-bX>0\}} - \mathbb{1}_{\{zY-bX>0\}}}{h_n}\right] = 0. \tag{A.2}$$

To see this, we start by noticing that

$$\mathbb{E}\left[g(bX, zY)\, \frac{\mathbb{1}_{\{(z+h_n)Y-bX>0\}} - \mathbb{1}_{\{zY-bX>0\}}}{h_n}\right] = \int_{\mathbb{R}} f(y)\, \frac{\operatorname{sgn}(h_n y)}{h_n} \left(\int_{I_n^y} g(bx, zy)\, f(x)\; dx\right) dy, \tag{A.3}$$

where $\operatorname{sgn}(\cdot)$ is the signum function,[9] and I_n^y is the interval of all real numbers between zy/b and $(z + h_n)\, y/b$. By Lebesgue's differentiation theorem, almost every real number belongs to the set $\mathscr{L}_f$ of Lebesgue points of f,[10] which in turn implies that $N_z := \left\{\frac{by}{z} : \ y \notin \mathscr{L}_f\right\}$ has Lebesgue measure zero. Next, fix an arbitrary $\varepsilon > 0$, and let y outside of $N_z \cup \{0\}$. It follows from the continuity of $g(\cdot, \cdot)$ that there exists $\delta_1 > 0$ such that

$$g(bx, zy) = g(bx, zy) - g(zy, zy) < \frac{1}{2}$$

for all $x \in \mathbb{R}$ with $|x - zy/b| < \delta$. Moreover, since $zy/b \in \mathscr{L}_f$, there exists $\delta_2 > 0$ such that

$$\frac{1}{2h} \int_{\left(\frac{zy}{b}-h, \frac{zy}{b}+h\right)} \left| f(x) - f\left(\frac{zy}{b}\right)\right| dy < \varepsilon$$

for all $0 < h < \delta_2$. Consequently, for all n sufficiently large,

$$\frac{1}{|h_n y/b|} \int_{I_n^y} g(bx, zy)\, f(x)\; dx \le \frac{2}{2\,|h_n y/b|} \int_{\left(\frac{zy}{b}-\left|\frac{h_n y}{b}\right|, \frac{zy}{b}+\left|\frac{h_n y}{b}\right|\right)} g(bx, zy)\, f(x)\; dx < \varepsilon,$$

so the sequence of integrands of the outer integral on the right-hand side of (A.3) converges to 0 almost everywhere (as $n \to +\infty$). Finally, for all $n \in \mathbb{N}$ and all $y \in \mathbb{R}$,

$$f(y)\, \frac{1}{|h_n|} \int_{I_n^y} g(bx, zy)\, f(x)\; dx \le \frac{2}{|b|} f(y)\, |y|\, g(zy - |y|\,, zy)\, f^*\left(\frac{zy}{b}\right) \le \frac{2}{|b|} f(y)\, |y|^2\, \psi_2^z(y)\, f^*\left(\frac{zy}{b}\right),$$

where $f^*(\cdot)$ denotes the maximal function of $f(\cdot)$.[11] Moreover, there is $C_\epsilon > 0$ such that

$$\int_{\mathbb{R}} f(y)\, |y|^2\, \psi_2^z(y)\, f^*\left(\frac{zy}{b}\right) dy \\ \le \left(\int_{\mathbb{R}} \psi_2^z(y)^{1+\epsilon}\, |y|^{2(1+\epsilon)}\, f(y)\; dy\right)^{\frac{1}{1+\epsilon}} C_\epsilon \left|\frac{b}{z}\right|^{\frac{\epsilon}{2\epsilon+1}} \left(\int_{\mathbb{R}} f(y)^{\frac{2\epsilon+1}{\epsilon}}\; dy\right)^{\frac{\epsilon}{1+\epsilon}} < +\infty,$$

by Hölder's inequality together with the Hardy-Littlewood maximal inequality. Hence, (A.2) follows from the dominated convergence theorem. □

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[9] The *signum function* $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ is defined by

$$\operatorname{sgn}(x) := \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

[10] Recall that $x \in \mathbb{R}$ is a *Lebesgue point* of a locally Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ if

$$\lim_{h\to0^+} \frac{1}{2h} \int_{(x-h,x+h)} |f(y) - f(x)|\, dy = 0.$$

[11] Recall that, given a locally Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$, its *Hardy-Littlewood maximal function* $f^* : \mathbb{R} \to [-\infty, +\infty]$ is defined by

$$f^*(x) := \sup_{r>0} \frac{1}{2r} \int_{(x-r,x+r)} |f(t)|\; dt, \quad \text{for all } x \in \mathbb{R}.$$

The next lemma is, despite its mathematical simplicity, central to our study of personal equilibria. Parts *(i)* and *(ii)* imply that investors cannot achieve infinite bliss or grief from any combination of investment strategy and reference payoff. In *(iii)* and *(iv)*, we study some of the main properties (such as continuity and differentiability) of the reference-dependent utility function given the safe reference and a risky reference, respectively; these are needed, in particular, for obtaining the first-order conditions of Lemma A.3. Lastly, part *(v)* is used to determine the preferred personal equilibrium.

Lemma A.2. *For X and $u(\cdot)$ satisfying Assumption 2.1, define $\mathcal{U}:\ \mathbb{R}^2 \to [-\infty, +\infty]$ as*

$$\mathcal{U}(z,b) := \mathbb{E}[u(\widetilde{w_0} + zY) + v(u(\widetilde{w_0} + zY) - u(\widetilde{w_0} + bX))]\,, \quad \textit{for all } (z,b) \in \mathbb{R}^2.$$

(i) *For all $b \in \mathbb{R}$, there exists $C \equiv C(b) > 0$ such that $\mathcal{U}(z,b) \le C\,(1+|z|)$ for all $z \in \mathbb{R}$.*

(ii) *For all $b, z \in \mathbb{R}$, there exists $C' \equiv C'(z,b) > 0$ such that $\mathcal{U}(z,b) \ge -C'\,(1+|z|)$.*

(iii) *The function $\Xi_0:\ \mathbb{R} \to \mathbb{R}$ defined by*

$$\Xi_0(z) := \mathcal{U}(z,0)\,, \quad \textit{for all } z \in \mathbb{R},$$

is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$, with

$$\Xi_0'(z) = \begin{cases} \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + v_+'(u(\widetilde{w_0} + zY) - u(\widetilde{w_0})))\,Y\,\mathbb{1}_{\{Y<0\}}] \\ \quad + \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + v_-'(u(\widetilde{w_0}) - u(\widetilde{w_0} + zY)))\,Y\,\mathbb{1}_{\{Y>0\}}]\,, & \textit{if } z < 0, \\ \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + v_+'(u(\widetilde{w_0} + zY) - u(\widetilde{w_0})))\,Y\,\mathbb{1}_{\{Y>0\}}] \\ \quad + \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + v_-'(u(\widetilde{w_0}) - u(\widetilde{w_0} + zY)))\,Y\,\mathbb{1}_{\{Y<0\}}]\,, & \textit{if } z > 0. \end{cases} \tag{A.4}$$

Moreover,

$$\Xi_0'(0\pm) := \lim_{z\to 0^\pm} \frac{\Xi_0(z) - \Xi_0(0)}{z} = u'(\widetilde{w_0})\,[\mu_+\,(1 + v_\pm'(0)) - \mu_-\,(1 + v_\mp'(0))] = \lim_{z\to 0^\pm} \Xi_0'(z)\,. \tag{A.5}$$

(iv) *For all $b \in \mathbb{R} \setminus \{0\}$, the function $\Xi_b:\ \mathbb{R} \to \mathbb{R}$ defined by*

$$\Xi_b(z) := \mathcal{U}(z,b)\,, \quad \textit{for all } z \in \mathbb{R},$$

is differentiable on $\mathbb{R}$, with

$$\begin{aligned} \Xi_b'(z) = {} & \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + v_+'(u(\widetilde{w_0} + zY) - u(\widetilde{w_0} + bX)))\,Y\,\mathbb{1}_{\{zY - bX > 0\}}] \\ & + \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + v_-'(u(\widetilde{w_0} + bX) - u(\widetilde{w_0} + zY)))\,Y\,\mathbb{1}_{\{zY - bX < 0\}}]\,, \quad \textit{for all } z \in \mathbb{R}. \end{aligned} \tag{A.6}$$

(v) *If, in addition, Assumption 2.3 holds, then the function $\Pi:\ \mathbb{R} \to \mathbb{R}$ defined by*

$$\Pi(z) := \mathcal{U}(z,z) = \mathbb{E}[u(\widetilde{w_0} + zY)] - \frac{\eta\,(1-\lambda)}{2\,(1-\eta)}\mathbb{E}\Big[\big|u(\widetilde{w_0} + zY) - u(\widetilde{w_0} + zX)\big|\Big]\,, \quad \textit{for all } z \in \mathbb{R},$$

is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$, with

$$\Pi'(z) = \begin{cases} \left(1 + \frac{\eta(1-\lambda)}{1-\eta}\right)\mathbb{E}[u'(\widetilde{w_0} + zY)\,Y] - \frac{2\eta(1-\lambda)}{1-\eta}\mathbb{E}[u'(\widetilde{w_0} + zY)\,Y\,\mathbb{1}_{\{Y - X < 0\}}]\,, & \textit{if } z < 0, \\ \left(1 + \frac{\eta(1-\lambda)}{1-\eta}\right)\mathbb{E}[u'(\widetilde{w_0} + zY)\,Y] - \frac{2\eta(1-\lambda)}{1-\eta}\mathbb{E}[u'(\widetilde{w_0} + zY)\,Y\,\mathbb{1}_{\{Y - X > 0\}}]\,, & \textit{if } z > 0. \end{cases}$$

Moreover,

$$\Pi'(0\pm) := \lim_{z\to 0^\pm} \frac{\Pi(z) - \Pi(0)}{z} = u'(\widetilde{w_0})\left(\mu \mp \frac{\eta\,(1-\lambda)}{2\,(1-\eta)}\,\Delta\right) = \lim_{z\to 0^\pm} \Pi'(z)\,, \tag{A.7}$$

where $\Delta := \mathbb{E}[|X - Y|]$ denotes the mean-absolute difference *of the distribution of X.*

Proof. For readability, we split the proof into several steps.

(i) Let $b \in \mathbb{R}$ be given. Since $u(\cdot)$ and $v_+(\cdot)$ are both concave, it is possible to find $C_1 > 0$ such that $u(x) \leq C_1\,(1+|x|)$ for all $x \in \mathbb{R}$, and $v_+(x) \leq C_1\,(1+x)$ for all $x \in [0,+\infty)$. These two inequalities combined with the assumption that $v_-(\cdot)$ is non-negative yield, for all $z \in \mathbb{R}$,

$$\mathscr{U}(z,b) \leq C_1\left(1+\left|\widetilde{w_0}\right| + |z|\,\mathbb{E}[|Y|]\right) + C_1\,\mathbb{E}[(1+u(\widetilde{w_0}+zY) - u(\widetilde{w_0}+bX))\,\mathbb{1}_{\{zY>bX\}}]\,.$$

Next, fix an arbitrary $z \in \mathbb{R}$. It follows from the fact that $u'(\cdot)$ is both non-increasing and strictly positive that, for each ω outside of the null set $N := \{|X| = +\infty\} \cup \{|Y| = +\infty\}$,

$$(u(\widetilde{w_0}+zY(\omega)) - u(\widetilde{w_0}+bX(\omega)))\,\mathbb{1}_{\{zY>bX\}}(\omega) \leq u'(\widetilde{w_0}+bX(\omega))\,|zY(\omega) - bX(\omega)|\,.$$

As X and Y are independent,

$$\mathbb{E}[u'(\widetilde{w_0}+bX)\,|zY-bX|] \leq |z|\,\mathbb{E}[u'(\widetilde{w_0}+bX)]\,\mathbb{E}[|Y|] + |b|\,\mathbb{E}[u'(\widetilde{w_0}+bX)\,|X|]\,.$$

Hence, choose

$$C := C_1\max\left\{2+\left|\widetilde{w_0}\right| + |b|\,\mathbb{E}[u'(\widetilde{w_0}+bX)\,|X|]\,,\mathbb{E}[|Y|]\,(1+\mathbb{E}[u'(\widetilde{w_0}+bX)])\right\},$$

which is finite in virtue of Assumption 2.1, strictly positive, and independent of z. Note that $\mathscr{U}(z,b) < +\infty$ for all $(z,b) \in \mathbb{R}^2$.

(ii) Let $b, z \in \mathbb{R}$ be arbitrary, but fixed. Because the function $v_-(\cdot)$ is concave, there exists some $C_2 > 0$ such that $v_-(x) \leq C_2\,(1+x)$ for all $x \in [0,+\infty)$, which together with $v_+(x) \geq 0$ for all $x \in [0,+\infty)$ leads to

$$\mathscr{U}(z,b) \geq \mathbb{E}[u(\widetilde{w_0}+zY)] - C_2\,\mathbb{E}[(1+u(\widetilde{w_0}+bX) - u(\widetilde{w_0}+zY))\,\mathbb{1}_{\{zY<bX\}}]\,.$$

Arguing as in *(i)*, $\mathscr{U}(z,b) \geq -C'\,(1+|z|)$, where

$$C' := \max\left\{\left|u(\widetilde{w_0})\right| + C_2\,(1+|b|\,\mathbb{E}[|X|]\,\mathbb{E}[u'(\widetilde{w_0}+zY)])\,,(1+C_2)\,\mathbb{E}[u'(\widetilde{w_0}+zY)\,|Y|]\right\}.$$

In particular, $\mathscr{U}(z,b) > -\infty$ for all $(z,b) \in \mathbb{R}^2$.

(iii) We show separately that $\Xi_0(\cdot)$ is continuous at $z=0$, and differentiable on $\mathbb{R}\setminus\{0\}$ with left and right derivatives at $z=0$.

(a) Let $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence converging to zero. Without loss of generality, assume that $|z_n| \leq 1$ for all $n \in \mathbb{N}$. The continuity of both $u(\cdot)$ and $v(\cdot)$ implies that the sequence

$$\{u(\widetilde{w_0}+z_nY) + v(u(\widetilde{w_0}+z_nY) - u(\widetilde{w_0}))\}_{n\in\mathbb{N}}$$

converges a.s. (as $n \to +\infty$) to $u(\widetilde{w_0})$. In addition, it is dominated a.s. by the integrable random variable

$$G := C_1 + C_2 + \left|u(\widetilde{w_0})\right| + C_1 u'(\widetilde{w_0})\,|Y| + (1+C_2)\,u'(\widetilde{w_0}-|Y|)\,|Y|\,,$$

where C_1 and C_2 are the strictly positive constants obtained in steps *(i)* and *(ii)*. Hence,

$$\lim_{n\to+\infty}\Xi_0(z_n) = \mathbb{E}[u(\widetilde{w_0})] = \Xi_0(0)\,.$$

(b) To see that $\Xi_0(\cdot)$ is differentiable on $(0,+\infty)$, let $z>0$ be given, and consider an arbitrary sequence $\{z_n\}_{n\in\mathbb{N}}$ of real numbers different from z and converging to z. Assume further that every z_n is strictly between $\frac{z}{2}$ and $\frac{3z}{2}$. Then, for all $n\in\mathbb{N}$,

$$\begin{aligned}\frac{\Xi_0(z_n)-\Xi_0(z)}{z_n-z} &= \mathbb{E}\left[\frac{u(\widetilde{w_0}+z_nY)-u(\widetilde{w_0}+zY)}{z_n-z}\right]\\ &+\mathbb{E}\left[\frac{\nu_+(u(\widetilde{w_0}+z_nY)-u(\widetilde{w_0}))-\nu_+(u(\widetilde{w_0}+zY)-u(\widetilde{w_0}))}{z_n-z}\mathbb{1}_{\{Y>0\}}\right]\\ &-\mathbb{E}\left[\frac{\nu_-(u(\widetilde{w_0})-u(\widetilde{w_0}+z_nY))-\nu_-(u(\widetilde{w_0})-u(\widetilde{w_0}+zY))}{z_n-z}\mathbb{1}_{\{Y<0\}}\right].\end{aligned}\tag{A.8}$$

The sequence

$$\left\{\frac{u(\widetilde{w_0}+z_nY)-u(\widetilde{w_0}+zY)}{z_n-z}\right\}_{n\in\mathbb{N}}$$

converges a.s. to $u'(\widetilde{w_0}+zY)\,Y$, and is dominated a.s. by the random variable $G_1:=u'\Big(\widetilde{w_0}-\frac{3z}{2}|Y|\Big)|Y|$, which is integrable because (recall Assumption 2.1*(ii)* and Hölder's inequality)

$$\begin{aligned}&\mathbb{E}\left[u'\left(\widetilde{w_0}-\frac{3z}{2}|Y|\right)|Y|\right]\\ &\quad\le\mathbb{E}\left[u'\left(\widetilde{w_0}-\frac{3z}{2}|Y|\right)\right]^{\frac{1}{2}}\mathbb{E}\left[u'\left(\widetilde{w_0}-\frac{3z}{2}|Y|\right)^{1+\epsilon}|Y|^{2+2\epsilon}\right]^{\frac{1}{2(1+\epsilon)}}\mathbb{E}\left[1^{\frac{2(1+\epsilon)}{\epsilon}}\right]^{\frac{\epsilon}{2(1+\epsilon)}}<+\infty.\end{aligned}$$

Thus, the first expectation on the right-hand side of (A.8) tends to $\mathbb{E}[u'(\widetilde{w_0}+zY)\,Y]$ as $n\to+\infty$. Two more applications of the dominated convergence theorem, with dominating random variables $G_2:=\nu_+'(0)\,u'\Big(\widetilde{w_0}+\frac{z}{2}Y\Big)|Y|$ and $G_3:=\nu_-'(0)\,u'\Big(\widetilde{w_0}+\frac{3z}{2}Y\Big)|Y|$, give that the second and third expectations on the right-hand side of (A.8) have limits

$$\mathbb{E}[\nu_+'(u(\widetilde{w_0}+zY)-u(\widetilde{w_0}))\,u'(\widetilde{w_0}+zY)\,Y\mathbb{1}_{\{Y>0\}}]$$

and

$$-\mathbb{E}[\nu_-'(u(\widetilde{w_0})-u(\widetilde{w_0}+zY))\,u'(\widetilde{w_0}+zY)\,Y\mathbb{1}_{\{Y<0\}}],$$

respectively. Hence, the difference quotients in (A.8) tend (as $n\to+\infty$) to the second expression given in (A.4). Differentiability of $\Xi_0(\cdot)$ on $(-\infty,0)$ follows similarly.

(c) Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of strictly negative real numbers convergent to zero. Also, $z_n>-1$ for all $n\in\mathbb{N}$. By *(b)*,

$$\begin{aligned}\Xi_0'(z_n)&=\mathbb{E}[u'(\widetilde{w_0}+z_nY)\,(1+\nu_+'(u(\widetilde{w_0}+z_nY)-u(\widetilde{w_0})))\,Y\mathbb{1}_{\{Y<0\}}]\\ &\quad+\mathbb{E}[u'(\widetilde{w_0}+z_nY)\,(1+\nu_-'(u(\widetilde{w_0})-u(\widetilde{w_0}+z_nY)))\,Y\mathbb{1}_{\{Y>0\}}]\end{aligned}$$

for all $n\in\mathbb{N}$. Using the dominated convergence theorem twice, we derive the last equality in (A.5). The non-differentiability of $\Xi_0(\cdot)$ at $z=0$ is evident upon observing

$$\Xi_0'(0+)-\Xi_0'(0-)=u'(\widetilde{w_0})\,(\mu_++\mu_-)\,[\nu_+'(0)-\nu_-'(0)]<0.$$

(iv) Let $b\in\mathbb{R}\setminus\{0\}$. An analogous argument to the one given in *(iii)(a)* establishes the continuity of $\Xi_b(\cdot)$ at $z=0$. We show below that $\Xi_b(\cdot)$ is differentiable.

(a) With D as in Lemma A.1, define $g_1:\mathbb{R}^2\to\mathbb{R}$ and $g_2:\mathbb{R}^2\to\mathbb{R}$ respectively as

$$g_1(x,y):=\begin{cases}\nu_+(u(\widetilde{w_0}+y)-u(\widetilde{w_0}+x)), & \text{if } (x,y)\in D,\\ 0, & \text{otherwise,}\end{cases}$$

$$g_2(x,y):=\begin{cases}\nu_-(u(\widetilde{w_0}-x)-u(\widetilde{w_0}-y)), & \text{if } (x,y)\in D,\\ 0, & \text{otherwise,}\end{cases}$$

and denote by $\Gamma_i(\cdot)$ the function defined by (A.1) associated with $g_i(\cdot,\cdot)$, for $i \in \{1,2\}$. Straightforward computations show, for all $z \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$, and $0 < |h| < 1$,

$$\left|\frac{g_1(bx,(z+h)\,y) - g_1(bx,zy)}{h}\right| \mathbb{1}_{\{(z+h)y-bx>0\}} \leq \nu'_+(0)\,u'(\widetilde{w_0} - (|z|+1)\,|y|)\,|y|\,,$$
$$\left|\frac{g_2(bx,(z+h)\,y) - g_2(bx,zy)}{h}\right| \mathbb{1}_{\{(z+h)y-bx>0\}} \leq \nu'_-(0)\,u'(\widetilde{w_0} - (|z|+1)\,|y|)\,|y|\,.$$

Furthermore,

$$g_1(zy-|y|\,,zy) \leq \nu'_+(0)\,u'(\widetilde{w_0} - (|z|+1)\,|y|)\,|y|\,,$$
$$g_2(zy-|y|\,,zy) \leq \nu'_-(0)\,u'(\widetilde{w_0} - (|z|+1)\,|y|)\,|y|\,.$$

Since $\Xi_b(z) = \mathbb{E}[u(\widetilde{w_0}+zY)] + \Gamma_1(b,z) - \Gamma_2(-b,-z)$ for all $z \in \mathbb{R}$, it follows from (A.1) that $\Xi'_b(z)$ is given by the expression on the right-hand side of (A.6) for all $z \in \mathbb{R}\setminus\{0\}$.

(b) By the result in *(a)* and the dominated convergence theorem,

$$\lim_{z\to 0}\Xi'_b(z) = u'(\widetilde{w_0})\,\mu\mathbb{E}[(1+\nu'_+(u(\widetilde{w_0}) - u(\widetilde{w_0}+bX)))\,\mathbb{1}_{\{bX<0\}}]$$
$$+\,u'(\widetilde{w_0})\,\mu\mathbb{E}[(1+\nu'_-(u(\widetilde{w_0}+bX) - u(\widetilde{w_0})))\,\mathbb{1}_{\{bX>0\}}]\,.$$

Hence, $\Xi_b(\cdot)$ is not only continuous but differentiable at $z=0$. Also, note that $\Xi'_b(0)$ has the same sign as μ.

(v) The proof that $\Pi(\cdot)$ is continuous at $z=0$ and differentiable on $\mathbb{R}\setminus\{0\}$ follows similar steps to those of *(iii)(a)* and *(iii)(b)*, so we omit the details. Another application of the dominated convergence theorem shows that the limits $\Pi'(0\pm)$ exist and are given by (A.7). Finally,

$$\Pi'(0+) - \Pi'(0-) = -u'(\widetilde{w_0})\,\Delta\,\frac{\eta\,(1-\lambda)}{1-\eta} < 0,$$

thus $\Pi(\cdot)$ is not differentiable at $z=0$. □

The following lemma establishes necessary first-order conditions for a portfolio to be a personal equilibrium. These conditions are sufficient provided that we specify an additional relation between $u(\cdot)$ and $\nu_-(\cdot)$ (recall Assumption 3.9 above). It was shown by Kőszegi and Rabin (2007, Proposition 11(i)) that the safe portfolio is a personal equilibrium only if (3.3) holds.

LEMMA A.3. *Let Assumption 2.1 hold.*

(i) *If* $0 \in PE(w_0)$, *then*
$$\frac{1+\nu'_+(0)}{1+\nu'_-(0)} \leq \frac{\mu_+}{\mu_-} \leq \frac{1+\nu'_-(0)}{1+\nu'_+(0)}. \tag{3.3}$$
Conversely, if in addition to (3.3) *Assumption 3.9 holds, then* $0 \in PE(w_0)$.

(ii) *For any risky portfolio* ϕ, *if* $\phi \in PE(w_0)$, *then*

$$\mathbb{E}\Big[u'(\widetilde{w_0}+\phi Y)\,(1+\nu'_+(u(\widetilde{w_0}+\phi Y) - u(\widetilde{w_0}+\phi X)))\,Y\,\mathbb{1}_{\{\phi(Y-X)>0\}}\Big]$$
$$= -\mathbb{E}\Big[u'(\widetilde{w_0}+\phi Y)\,(1+\nu'_-(u(\widetilde{w_0}+\phi X) - u(\widetilde{w_0}+\phi Y)))\,Y\,\mathbb{1}_{\{\phi(Y-X)<0\}}\Big]\,. \tag{A.9}$$

Conversely, if in addition to (A.9) *Assumption 3.9 holds, then* $\phi \in PE(w_0)$.

Proof. By Lemma A.2, $\Xi'_b(b)$ exists for all $b \in \mathbb{R}\setminus\{0\}$. As the left and right derivatives of $\Xi_0(\cdot)$ differ at 0, the safe portfolio must be dealt with separately from the risky portfolios.

(i) If (3.3) fails, then either $\Xi_0'(0+) > 0$ (thus $U\Big(V_T^{\varphi}\Big|\, V_T^0\Big) > U\Big(V_T^0\Big|\, V_T^0\Big)$ for some $\varphi > 0$) or $\Xi_0'(0-) < 0$ (thus $U\Big(V_T^{\varphi}\Big|\, V_T^0\Big) > U\Big(V_T^0\Big|\, V_T^0\Big)$ for some $\varphi < 0$), where $\Xi_0(\cdot)$ is the function of Lemma A.2. Hence, the safe portfolio is not optimal given the safe reference.

On the other hand, assume that (3.3) and (3.9) both hold. Then $\Xi_0(\cdot)$ has a (relative) maximum at $z = 0$. Moreover, for all $z_1, z_2 \in (-\infty, 0]$ and $\vartheta \in [0, 1]$,

$$\begin{aligned}
&\Xi_0(\vartheta z_1 + (1-\vartheta)\, z_2) \\
&\geq \vartheta \mathbb{E}[u(\widetilde{w_0} + z_1 Y)\, \mathbb{1}_{\{Y<0\}}] + (1-\vartheta)\, \mathbb{E}[u(\widetilde{w_0} + z_2 Y)\, \mathbb{1}_{\{Y<0\}}] \\
&\quad + \vartheta \mathbb{E}[v_+(u(\widetilde{w_0} + z_1 Y) - u(\widetilde{w_0}))\, \mathbb{1}_{\{Y<0\}}] + (1-\vartheta)\, \mathbb{E}[v_+(u(\widetilde{w_0} + z_2 Y) - u(\widetilde{w_0}))\, \mathbb{1}_{\{Y<0\}}] \\
&\quad + \vartheta \mathbb{E}[(u(\widetilde{w_0} + z_1 Y) - v_-(u(\widetilde{w_0}) - u(\widetilde{w_0} + z_1 Y)))\, \mathbb{1}_{\{Y>0\}}] \\
&\quad + (1-\vartheta)\, \mathbb{E}[(u(\widetilde{w_0} + z_2 Y) - v_-(u(\widetilde{w_0}) - u(\widetilde{w_0} + z_2 Y)))\, \mathbb{1}_{\{Y>0\}}] \\
&= \vartheta \Xi_0(z_1) + (1-\vartheta)\, \Xi_0(z_2)\,.
\end{aligned}$$

where we use that $u(\cdot)$ is concave, $v_+(\cdot)$ is strictly increasing and concave, and the mapping $x \mapsto u(x) - v_-(u(y) - u(x))$ is concave on $(-\infty, y]$ for every fixed $y \in \mathbb{R}$. Hence, $\Xi_0(\cdot)$ is concave on the half-interval $(-\infty, 0]$. Likewise, $\Xi_0(\cdot)$ is concave on $[0, +\infty)$.

(ii) Let $\phi \neq 0$. If $\Xi_\phi(\cdot)$ attains its absolute maximum at $z = \phi$, then $\Xi_\phi'(\phi) = 0$. Conversely, assume that (A.9) and (3.9) both hold. For all $z_1, z_2 \in \mathbb{R}$ such that $z_1 < z_2$,

$$\begin{aligned}
\Xi_\phi'(z_1) \geq\; &\mathbb{E}\Big[u'(\widetilde{w_0} + z_2 Y)\Big(1 + v_+'(u(\widetilde{w_0} + z_2 Y) - u(\widetilde{w_0} + \phi X))\Big) Y\, \mathbb{1}_{\{Y<0, \phi X < z_2 Y\}}\Big] \\
&+ \mathbb{E}\Big[u'(\widetilde{w_0} + z_1 Y)\Big(1 + v_+'(u(\widetilde{w_0} + z_1 Y) - u(\widetilde{w_0} + \phi X))\Big) Y\, \mathbb{1}_{\{Y<0, z_2 Y < \phi X < z_1 Y\}}\Big] \\
&+ \mathbb{E}\Big[u'(\widetilde{w_0} + z_2 Y)\Big(1 + v_+'(u(\widetilde{w_0} + z_2 Y) - u(\widetilde{w_0} + \phi X))\Big) Y\, \mathbb{1}_{\{Y>0, \phi X < z_1 Y\}}\Big] \\
&+ \mathbb{E}\Big[u'(\widetilde{w_0} + z_2 Y)\Big(1 + v_-'(u(\widetilde{w_0} + \phi X) - u(\widetilde{w_0} + z_2 Y))\Big) Y\, \mathbb{1}_{\{Y<0, z_1 Y < \phi X\}}\Big] \\
&+ \mathbb{E}\Big[u'(\widetilde{w_0} + z_1 Y)\Big(1 + v_-'(u(\widetilde{w_0} + \phi X) - u(\widetilde{w_0} + z_1 Y))\Big) Y\, \mathbb{1}_{\{Y>0, z_1 Y < \phi X < z_2 Y\}}\Big] \\
&+ \mathbb{E}\Big[u'(\widetilde{w_0} + z_2 Y)\Big(1 + v_-'(u(\widetilde{w_0} + \phi X) - u(\widetilde{w_0} + z_2 Y))\Big) Y\, \mathbb{1}_{\{Y>0, z_2 Y < \phi X\}}\Big]\,,
\end{aligned}$$

as $u(\cdot)$ and $v_+(\cdot)$ are both strictly increasing and concave, and $x \mapsto u(x) - v_-(u(y) - u(x))$ is decreasing on $(-\infty, y]$ for every fixed $y \in \mathbb{R}$. Finally, combining the above inequality with

$$\begin{aligned}
&\mathbb{E}\Big[u'(\widetilde{w_0} + z_1 Y)\Big(1 + v_+'(u(\widetilde{w_0} + z_1 Y) - u(\widetilde{w_0} + \phi X))\Big) Y\, \mathbb{1}_{\{Y<0, z_2 Y < \phi X < z_1 Y\}}\Big] \\
&\geq \mathbb{E}\Big[u'(\widetilde{w_0} + \phi X)\Big(1 + v_+'(0)\Big) Y\, \mathbb{1}_{\{Y<0, z_2 Y < \phi X < z_1 Y\}}\Big] \geq \mathbb{E}\Big[u'(\widetilde{w_0} + \phi X)\Big(1 + v_-'(0)\Big) Y\, \mathbb{1}_{\{Y<0, z_2 Y < \phi X < z_1 Y\}}\Big] \\
&\geq \mathbb{E}\Big[u'(\widetilde{w_0} + z_2 Y)\Big(1 + v_-'(u(\widetilde{w_0} + \phi X) - u(\widetilde{w_0} + z_2 Y))\Big) Y\, \mathbb{1}_{\{Y<0, z_2 Y < \phi X < z_1 Y\}}\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[u'(\widetilde{w_0} + z_1 Y)\Big(1 + v_-'(u(\widetilde{w_0} + \phi X) - u(\widetilde{w_0} + z_1 Y))\Big) Y\, \mathbb{1}_{\{Y>0, z_1 Y < \phi X < z_2 Y\}}\Big] \\
&\geq \mathbb{E}\Big[u'(\widetilde{w_0} + \phi X)\Big(1 + v_-'(0)\Big) Y\, \mathbb{1}_{\{Y>0, z_1 Y < \phi X < z_2 Y\}}\Big] \geq \mathbb{E}\Big[u'(\widetilde{w_0} + \phi X)\Big(1 + v_+'(0)\Big) Y\, \mathbb{1}_{\{Y>0, z_1 Y < \phi X < z_2 Y\}}\Big] \\
&\geq \mathbb{E}\Big[u'(\widetilde{w_0} + z_2 Y)\Big(1 + v_+'(u(\widetilde{w_0} + z_2 Y) - u(\widetilde{w_0} + \phi X))\Big) Y\, \mathbb{1}_{\{Y>0, z_1 Y < \phi X < z_2 Y\}}\Big]
\end{aligned}$$

yields $\Xi_b'(z_1) \geq \Xi_b'(z_2)$. $\square$

The result below is the key to determining risky personal equilibria and deriving their properties. We introduce the auxiliary function $\Lambda(\cdot)$ defined by $\Lambda(\phi) = \Xi_\phi'(\phi)$ for all $\phi \in \mathbb{R} \setminus \{0\}$, so that $\Lambda(\phi) = 0$ effectively corresponds to the first-order condition (A.9) derived in the preceding lemma. In other words, the only candidates for risky personal equilibria are precisely the roots of $\Lambda(\cdot)$. Moreover, by studying the continuity, monotonicity, and limiting behavior of $\Lambda(\cdot)$, we can determine the number of risky personal equilibria as well as their relative location.

LEMMA A.4. *Let Assumption 2.1 hold. Define* $\Lambda : \mathbb{R} \to \mathbb{R}$ *as*

$$\Lambda(z) = \begin{cases} \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + \nu_-'(u(\widetilde{w_0} + zX) - u(\widetilde{w_0} + zY)))\,Y\mathbb{1}_{\{Y-X>0\}}] \\ \quad + \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + \nu_+'(u(\widetilde{w_0} + zY) - u(\widetilde{w_0} + zX)))\,Y\mathbb{1}_{\{Y-X<0\}}]\,, & \text{if } z < 0, \\ 0, & \text{if } z = 0, \\ \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + \nu_-'(u(\widetilde{w_0} + zX) - u(\widetilde{w_0} + zY)))\,Y\mathbb{1}_{\{Y-X<0\}}] \\ \quad + \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + \nu_+'(u(\widetilde{w_0} + zY) - u(\widetilde{w_0} + zX)))\,Y\mathbb{1}_{\{Y-X>0\}}]\,, & \text{if } z > 0. \end{cases} \tag{A.10}$$

(i) *The function* $\Lambda(\cdot)$ *is continuous on* $\mathbb{R} \setminus \{0\}$, *with*

$$\Lambda(0\pm) \coloneqq \lim_{z\to 0^\pm} \Lambda(z) = u'(\widetilde{w_0})\,\mu\,(1 + \nu_-'(0)) - u'(\widetilde{w_0})\,\nu_-'(0)\,(1-\lambda)\,\frac{2\mu \pm \Delta}{4}. \tag{A.11}$$

(ii) *If* $u(\cdot)$ *is linear, then*

$$\Lambda(\pm\infty) \coloneqq \lim_{z\to \pm\infty} \Lambda(z) = u'(\widetilde{w_0})\,\mu\,(1 + \nu_-'(+\infty)) - u'(\widetilde{w_0})\,(\nu_-'(+\infty) - \nu_+'(+\infty))\,\frac{2\mu \pm \Delta}{4}.$$

If, in addition, Assumption 2.3 holds, then $\Lambda(\cdot)$ *is constant on each of the intervals* $(-\infty, 0)$ *and* $(0, +\infty)$.

(iii) *If* $u(\cdot)$ *is strictly concave with* $u'(-\infty) = +\infty$, *then* $\Lambda(\pm\infty) = \mp\infty$. *If, in addition, Assumption 2.3 holds, then* $\Lambda(\cdot)$ *is strictly decreasing on each of the intervals* $(-\infty, 0)$ *and* $(0, +\infty)$.

Proof. Define the functions $\Theta_\pm : \mathbb{R} \to \mathbb{R}$ and $\bar{\Theta}_\pm : \mathbb{R} \to \mathbb{R}$ as, respectively,

$$\Theta_\pm(z) \coloneqq \mathbb{E}[u'(\widetilde{w_0} + zY)\,Y\mathbb{1}_{\{\pm(Y-X)>0\}}]\,, \quad \text{for all } z \in \mathbb{R},$$

$$\bar{\Theta}_\pm(z) \coloneqq \begin{cases} \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + \nu_\pm'(u(\widetilde{w_0} + zY) - u(\widetilde{w_0} + zX)))\,Y\mathbb{1}_{\{\mp(Y-X)>0\}}]\,, & \text{if } z < 0, \\ 0, & \text{if } z = 0, \\ \mathbb{E}[u'(\widetilde{w_0} + zY)\,(1 + \nu_\pm'(u(\widetilde{w_0} + zY) - u(\widetilde{w_0} + zX)))\,Y\mathbb{1}_{\{\pm(Y-X)>0\}}]\,, & \text{if } z > 0. \end{cases}$$

Note that, for all $z \in \mathbb{R}$, independence of X and Y gives $\Theta_-(z) = \mathbb{E}[u'(\widetilde{w_0} + zY)\,Y\,(1 - F(Y))]$, where $F(\cdot)$ denotes the cumulative distribution function of X. In particular, this implies

$$\Theta_-(z) < (1 - F(0))\,\mathbb{E}[u'(\widetilde{w_0} + zY)\,Y] \leq 0 \tag{A.12}$$

for all $z \geq \theta$, with θ the unique solution of (3.6).

(i) The idea to show that $\Theta_\pm(\cdot)$ are continuous on $\mathbb{R}$ and $\bar{\Theta}_\pm(\cdot)$ are continuous on $\mathbb{R} \setminus \{0\}$ is to apply the dominated convergence theorem (we omit the proof, as it is a variation of the proof given above). Thus, it follows immediately from

$$\Lambda(z) = \bar{\Theta}_-(z) + \bar{\Theta}_+(z) \quad \text{for all } z \in \mathbb{R}$$

that $\Lambda(\cdot)$ is continuous on $\mathbb{R}\setminus\{0\}$. The proof of (A.11) uses equation (2.2) together with $\mathbb{E}[Y\mathbb{1}_{\{\pm(Y-X)>0\}}] = (2\mu \pm \Delta)/4$, and follows along the same lines of the proof of (A.5).

(ii) Suppose that $u(\cdot)$ is linear. Since $\Theta_\pm(z) = \Theta_\pm(0) = u'(\widetilde{w_0})\,\mathbb{E}[Y\mathbb{1}_{\{\pm(Y-X)>0\}}]$ for all $z \in \mathbb{R}$, it is trivial that $\lim_{z\to-\infty} \Theta_\pm(z) = \lim_{z\to+\infty} \Theta_\pm(z) = \Theta_\pm(0)$. Furthermore,

$$\bar{\Theta}_+(z) = \begin{cases} u'(\widetilde{w_0})\,\mathbb{E}[(1 + \nu_+'(u'(\widetilde{w_0})\,(Y-X)\,z))\,Y\mathbb{1}_{\{Y-X<0\}}]\,, & \text{if } z < 0, \\ u'(\widetilde{w_0})\,\mathbb{E}[(1 + \nu_+'(u'(\widetilde{w_0})\,(Y-X)\,z))\,Y\mathbb{1}_{\{Y-X>0\}}]\,, & \text{if } z > 0, \end{cases}$$

so using the dominated convergence theorem twice with the dominating random variable $(1 + \nu_+'(0))\,|Y|$ yields $\lim_{z\to\pm\infty} \bar{\Theta}_+(z) = (1 + \nu_+'(+\infty))\,\Theta_\pm(0)$. Using an analogous dominating argument, $\lim_{z\to\pm\infty} \bar{\Theta}_-(z) = (1 + \nu_-'(+\infty))\,\Theta_\mp(0)$. If, in addition, Assumption 2.3 holds, then

$$(1-\eta)\,\Lambda(z) = \begin{cases} \Theta_+(0) + (1 - \eta\,(1-\lambda))\,\Theta_-(0)\,, & \text{if } z < 0, \\ \Theta_-(0) + (1 - \eta\,(1-\lambda))\,\Theta_+(0)\,, & \text{if } z > 0. \end{cases}$$

(iii) Suppose that $u'(\cdot)$ is strictly decreasing with $u'(-\infty) = +\infty$. Each of the events $\{Y < 0, Y - X < 0\}$ and $\{Y > 0, Y - X > 0\}$ has strictly positive probability, since the monotonicity of the probability measure together with the independence and identical distribution of the random variables X and Y yields

$$\mathbb{P}\{Y < 0, Y - X < 0\} \geq \mathbb{P}\{Y < 0, X > 0\} = \mathbb{P}\{X < 0\}\,\mathbb{P}\{X > 0\},$$
$$\mathbb{P}\{Y > 0, Y - X > 0\} \geq \mathbb{P}\{Y > 0, X < 0\} = \mathbb{P}\{X > 0\}\,\mathbb{P}\{X < 0\}.$$

On the other hand, $\lim_{n\to+\infty} \mathbb{P}\left\{X < -\frac{1}{n}\right\} = \mathbb{P}\{X < 0\} > 0$ by continuity from below, thus the set $S := \{y \in (-\infty, 0) : \mathbb{P}\{X < y\} > 0\}$ is non-empty. Note that $(m, 0) \subseteq S$, where $m := \inf(S) \in [-\infty, 0)$. Moreover, if $m > -\infty$, then $\mathbb{P}\{X < m\} = \lim_{n\to+\infty} \mathbb{P}\left\{X < m - \frac{1}{n}\right\} = 0$. Consequently,

$$\mathbb{P}\{Y < 0, Y - X > 0\} = \int_m^0 \mathbb{P}\{X < y\}\; d\mathbb{P}_Y(y) > 0.$$

Likewise, $\mathbb{P}\{Y > 0, Y - X < 0\} > 0$. We complete the proof in the following steps.

(a) Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers diverging to $+\infty$. Without loss of generality, $z_n > 1$ for all $n \in \mathbb{N}$. Since the sequence $\{u'(\widetilde{w_0} + z_n Y)\, Y \mathbb{1}_{\{Y>0,\pm(Y-X)>0\}}\}_{n\in\mathbb{N}}$ converges a.s. to $u'(+\infty)\, Y \mathbb{1}_{\{Y>0,\pm(Y-X)>0\}}$, and is dominated by the integrable random variable $u'(\widetilde{w_0} + Y)\,|Y|$, we can use the dominated convergence theorem to obtain $\lim_{n\to+\infty} \mathbb{E}[u'(\widetilde{w_0} + z_n Y)\, Y \mathbb{1}_{\{Y>0,\pm(Y-X)>0\}}] = u'(+\infty)\,\mathbb{E}[Y \mathbb{1}_{\{Y>0,\pm(Y-X)>0\}}] < +\infty$. In addition, by Fatou's lemma,

$$\liminf_{n\to+\infty} \mathbb{E}[u'(\widetilde{w_0} + z_n Y)\,(-Y)\, \mathbb{1}_{\{Y<0,\pm(Y-X)>0\}}] \geq +\infty.$$

Hence,

$$\limsup_{n\to+\infty} \Theta_\pm(z_n) \leq \limsup_{n\to+\infty} \mathbb{E}[u'(\widetilde{w_0} + z_n Y)\, Y \mathbb{1}_{\{Y>0,\pm(Y-X)>0\}}] + \limsup_{n\to+\infty} \mathbb{E}[u'(\widetilde{w_0} + z_n Y)\, Y \mathbb{1}_{\{Y<0,\pm(Y-X)>0\}}] = -\infty.$$

A similar argument shows $\lim_{z\to-\infty} \Theta_\pm(z) = +\infty$.

(b) Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to+\infty} z_n = +\infty$ and $z_n > 1$ for all $n \in \mathbb{N}$. It follows from

$$\limsup_{n\to+\infty} \mathbb{E}[u'(\widetilde{w_0} + z_n Y)\,(1 + v'_+(u(\widetilde{w_0} + z_n Y) - u(\widetilde{w_0} + z_n X)))\, Y \mathbb{1}_{\{Y<0,Y-X>0\}}]$$
$$\leq (1 + v'_+(+\infty)) \limsup_{n\to+\infty} \mathbb{E}[u'(\widetilde{w_0} + z_n Y)\, Y \mathbb{1}_{\{Y<0,Y-X>0\}}] = -\infty$$

and

$$\limsup_{n\to+\infty} \mathbb{E}[u'(\widetilde{w_0} + z_n Y)\,(1 + v'_+(u(\widetilde{w_0} + z_n Y) - u(\widetilde{w_0} + z_n X)))\, Y \mathbb{1}_{\{Y>0,Y-X>0\}}]$$
$$\leq (1 + v'_+(0)) \limsup_{n\to+\infty} \mathbb{E}[u'(\widetilde{w_0} + z_n Y)\, Y \mathbb{1}_{\{Y>0,Y-X>0\}}] < +\infty$$

that $\lim_{z\to+\infty} \bar{\Theta}_+(z) = -\infty$. The proofs of $\lim_{z\to-\infty} \bar{\Theta}_+(z) = +\infty$ and $\lim_{z\to\pm\infty} \bar{\Theta}_-(z) = \mp\infty$ are analogous.

(c) By the results of *(a)* and *(b)*,

$$\limsup_{z\to+\infty} \Lambda(z) \leq \limsup_{z\to+\infty} \bar{\Theta}_-(z) + \limsup_{z\to+\infty} \bar{\Theta}_+(z) = -\infty,$$
$$\liminf_{z\to-\infty} \Lambda(z) \geq \liminf_{z\to-\infty} \bar{\Theta}_-(z) + \limsup_{z\to-\infty} \bar{\Theta}_+(z) = +\infty.$$

(d) Let, in addition, Assumption 2.3 hold. As the distribution of X is atomless and $u'(\cdot)$ is strictly decreasing, $\Theta_\pm(z_1) - \Theta_\pm(z_2) = \mathbb{E}[(u'(\widetilde{w_0} + z_1 Y) - u'(\widetilde{w_0} + z_2 Y))\, Y \mathbb{1}_{\{\pm(Y-X)>0\}}] > 0$ for all $z_1, z_2 \in \mathbb{R}$ such that $z_1 < z_2$. Hence,

$$\Lambda(z) = \begin{cases} \Theta_+(z) \,/\, (1-\eta) + (1 - \eta\,(1-\lambda))\,\Theta_-(z) \,/\, (1-\eta)\,, & \text{if } z < 0, \\ \Theta_-(z) \,/\, (1-\eta) + (1 - \eta\,(1-\lambda))\,\Theta_+(z) \,/\, (1-\eta)\,, & \text{if } z > 0, \end{cases}$$

is strictly decreasing on $\mathbb{R} \setminus \{0\}$. □

REMARK A.5. Suppose that $\mu > 0$, and let $\phi < 0$. As in the proof of Lemma A.3, $\Xi'_\phi(\cdot)$ is a decreasing function under (3.9), therefore

$$\Lambda(\phi) = \Xi'_\phi(\phi) \geq \Xi'_\phi(0) > 0. \tag{A.13}$$

This means that an investor expecting to short-sell the stock is actually better off buying a small amount of stock. Likewise, $\Lambda(\phi) = \Xi'_\phi(\phi) \leq \Xi'_\phi(0) < 0$ for all $\phi > 0$ if $\mu < 0$, meaning that a portfolio with a long position in the stock is never a personal equilibrium when the risk premium is negative.

The following lemma complements Remark 3.8 by providing an example where, even though preferences converge to linear, the results of Theorem 3.4 do not recover the ones in Theorem 3.1.

LEMMA A.6. *Let Assumption 2.3 hold, and fix $\eta, \lambda \in (0,1)$ such that (3.5) holds. If X has asymmetric Laplace distribution with scale parameter $\varsigma > 0$ and asymmetry parameter $\kappa \in (0,1)$, i.e.,*

$$f(x) = \frac{\varsigma}{\kappa + 1/\kappa} \begin{cases} e^{\varsigma x/\kappa} & , \text{if } x < 0, \\ e^{-\varsigma \kappa x} & , \text{if } x > 0, \end{cases}$$

then there exists a sequence of utility functions $\{u_n(\cdot)\}_{n\in\mathbb{N}}$ such that:

(i) *$u'_n(\cdot)$ is strictly decreasing with $u'_n(-\infty) = +\infty$ for all $n \in \mathbb{N}$;*

(ii) *$\lim_{n\to+\infty} u_n(x) = x$ for all $x \in \mathbb{R}$;*

(iii) *$\left\{\left|\theta^*_n\right|\right\}_{n\in\mathbb{N}}$ remains bounded, where each θ^*_n denotes the unique risky personal equilibrium for the reference-dependent problem (2.4) with utility $u_n(\cdot)$.*

Proof. First, note that

$$\mu = \frac{1-\kappa^2}{\varsigma\kappa} > 0.$$

Next, let $\varepsilon \in (0,1)$, and for all $n \in \mathbb{N}$ define the continuous and concave function $u_n : \mathbb{R} \to \mathbb{R}$ as

$$u_n := \begin{cases} -n + (1+\delta_n)\,(x+n)\,, & \text{if } x < -n, \\ x, & \text{if } x \in [-n, n]\,, \\ n + (1-\varepsilon)\,(x-n)\,, & \text{if } x > n, \end{cases}$$

where

$$\delta_n := \frac{\varepsilon\kappa^2 e^{-\varsigma n/\kappa}\left(\varsigma\kappa n + \kappa^2\right) + 1 - \kappa^4}{e^{-\varsigma\kappa n}\,(\varsigma\kappa n + 1)} > 0.$$

Note that $\lim_{n\to+\infty} \delta_n = +\infty$. Direct computations yield, for all $n \in \mathbb{N}$ and $z > 0$,

$$\mathbb{E}[u_n(zX)] = \frac{\varepsilon\kappa^2}{\varsigma\,(\kappa + 1/\kappa)} e^{-\frac{\varsigma n}{\kappa z}} z + \frac{1}{\varsigma\kappa^2\,(\kappa + 1/\kappa)}\left(1 - \kappa^4\right) z - \delta_n \frac{1}{\varsigma\kappa^2\,(\kappa + 1/\kappa)} e^{-\frac{\varsigma\kappa n}{z}} z,$$

therefore

$$\frac{d}{dz}\mathbb{E}[u_n(zX)] = \frac{\varepsilon\kappa^2}{\varsigma\,(\kappa + 1/\kappa)} e^{-\frac{\varsigma n}{\kappa z}} \left(\frac{\varsigma n}{\kappa z} + 1\right) + \frac{1}{\varsigma\kappa^2\,(\kappa + 1/\kappa)}\left(1 - \kappa^4\right) - \delta_n \frac{1}{\varsigma\kappa^2\,(\kappa + 1/\kappa)} e^{-\frac{\varsigma\kappa n}{z}} \left(\frac{\varsigma\kappa n}{z} + 1\right).$$

Since by construction of δ_n,

$$\begin{aligned}
&\frac{d}{dz}\mathbb{E}[u_n(zX)]\Big|_{z=1} \\
&\quad = \frac{\varepsilon\kappa^2}{\varsigma\,(\kappa+1/\kappa)}e^{-\frac{\varsigma n}{\kappa}}\left(\frac{\varsigma n}{\kappa}+1\right)+\frac{1}{\varsigma\kappa^2\,(\kappa+1/\kappa)}\left(1-\kappa^4\right)-\delta_n\frac{1}{\varsigma\kappa^2\,(\kappa+1/\kappa)}e^{-\varsigma\kappa n}\,(\varsigma\kappa n+1)=0,
\end{aligned}$$

we conclude that $\theta_n = 1$ is the unique classical utility maximizer for an investor with preferences given by $u_n(\cdot)$ and zero initial capital.

It is clear that $u_n(\cdot)$ is neither strictly concave nor differentiable; moreover, $u_n'(-\infty) = 1+\delta_n < +\infty$. Nevertheless, it is possible to locally regularize the function in such a way that it satisfies property *(i)* without affecting the optimizer θ_n. Then, $\theta_n^* > 0$ exists and is unique by Theorem 3.4, while (3.7) gives $\theta_n^* < \theta_n = 1$.

Hence, we have obtained a sequence $\{u_n(\cdot)\}_{n\in\mathbb{R}}$ converging pointwise to the (linear) identity function, whose corresponding sequence $\{\theta_n^*\}_{n\in\mathbb{N}}$ of risky personal equilibria is bounded above by 1. □

We conclude this subsection with alternative expressions for the Gini index which, in addition to being computationally convenient, facilitate the interpretation of our results.

Lemma A.7. *Let $\mu \neq 0$, and let Assumption 2.1 hold. Then,*

$$G = \frac{\Delta}{2\,|\mu|} = \frac{\mathbb{E}[X\vee Y]}{|\mu|} - 1 = 1 - \frac{\mathbb{E}[X\wedge Y]}{|\mu|}. \tag{A.14}$$

Moreover,

$$G \leq \frac{1}{|S|\,\sqrt{3}}.$$

Proof. Refer, e.g., to Yitzhaki and Schechtman (2013, Chapter 2). □

A.2 Proofs of Sections 3 and 4

Proof of Theorem 3.1. We prove the result for $\mu > 0$ only, as the other case is analogous. First, observe that $1-\eta\,(1-\lambda) < 1 < \mu_+/\mu_-$. Thus, in light of Lemma A.3, the safe portfolio is a personal equilibrium if and only if $\mu_+/\mu_- \leq 1/\,(1-\eta\,(1-\lambda))$. Turning to the risky personal equilibria, they can be characterized in terms of the roots of the function $\Lambda(\cdot)$ defined in Lemma A.4. Due to Remark A.5, no short-position portfolio can be a personal equilibrium. The remainder of the proof is carried out in three steps.

(i) Suppose that $G \leq 1$. Because $\Lambda(z) = \Lambda(0+) \geq u'(\widetilde{w_0})\,\mu\,(1-\eta\,(1-\lambda))\,/\,(1-\eta) > 0$ for all $z > 0$, there can be no personal equilibria with a long position in the stock either. Consequently, the safe portfolio is the only candidate for an equilibrium.

(ii) Suppose that $G > 1$. Clearly, $\Lambda(0+) \neq 0$ is equivalent to (3.2). Also, it follows from $\Delta \leq 2\,(\mu_+ + \mu_-)$ that (3.1) failing implies

$$1-\eta\,(1-\lambda) > \frac{\mu_-}{\mu_+} \geq \frac{\Delta-2\mu}{\Delta+2\mu} = \frac{G-1}{G+1}. \tag{A.15}$$

(iii) Suppose that (3.2) fails and consequently (3.1) holds, whence any long portfolio position (including zero) is a personal equilibrium. As $u(\cdot)$ is linear, the function $\Pi(\cdot)$ of Lemma A.2 is affine on each of the half-intervals $(-\infty, 0\,]$ and $[\,0, +\infty)$. In addition,

$$\Pi'(z) = \Pi'(0+) = u'(\widetilde{w_0})\,\mu\left(1-\frac{2G}{(1-\eta)\,(G+1)}\right) < 0$$

for all $z > 0$, so $\Pi(\cdot)$ attains its absolute maximum on $[\,0, +\infty)$ at $z = 0$. □

Proof of Theorem 3.4. Consider the case where $\mu > 0$ (a proof for $\mu < 0$ follows along the same lines), and let $\eta, \lambda \in (0,1)$ be given. By Lemma A.4, $\Lambda(\cdot)$ is continuous and strictly decreasing on $\mathbb{R} \setminus \{0\}$ with $\Lambda(\theta) = \eta\,(1-\lambda)\,\Theta_-(\theta)\,/\,(1-\eta) < 0$ (recall (A.12)). Moreover, the inequality in (A.13) shows that $\Lambda(\cdot)$ is strictly positive on $(-\infty, 0)$. We treat three cases.

(i) Suppose that $G \leq 1$. In this case $\Lambda(0+) > 0$, so $\Lambda(\cdot)$ takes the value zero at exactly one point $\theta^* \equiv \theta^*(\eta,\lambda) \in (0,\theta)$. Therefore, θ^* is the only personal equilibrium if (3.1) fails, otherwise $\mathrm{PE}(w_0) = \{0, \theta^*\}$.

(ii) Suppose that $G > 1$. It follows from (A.15) that (3.5) holds if (3.1) fails. Moreover, (3.5) is equivalent to $\Lambda(0+) > 0$. Then there are three possible scenarios: (3.5) fails, thus (3.1) holds, whence $\Lambda(\cdot)$ never attains zero (and 0 is the only personal equilibrium); or (3.1) fails, thus (3.5) holds, whence $\Lambda(\theta^*) = 0$ for some unique $\theta^* \equiv \theta^*(\eta,\lambda) \in (0,\theta)$; or (3.1) and (3.5) both hold, whence $\mathrm{PE}(w_0) = \{0, \theta^*\}$.

(iii) To show that 0 is the unique preferred personal equilibrium if (3.1), (3.5) and $G \geq 1$ are all binding, first note that $\psi(x) := \mathbb{E}[x \vee Y] \geq x \vee \mathbb{E}[Y]$ for all $x \in \mathbb{R}$, by Jensen's inequality. Therefore, $\mathbb{E}[X \vee Y] = \mathbb{E}[\mathbb{E}[X \vee Y | X]] = \mathbb{E}[\psi(X)] \geq \mathbb{E}[X \vee \mu] \geq \mathbb{E}[X \vee 0] = \mathbb{E}[X^+]$, which is equivalent to $G^{-1} \leq \mu/\mu_-$. This inequality together with (3.1) yields

$$\frac{\eta\,(1-\lambda)}{1-\eta} \geq \frac{1}{G}. \tag{A.16}$$

Finally, assume that $\Pi(\theta^*) \geq \Pi(0)$, with $\Pi(\cdot)$ as in Lemma A.2. It follows from $\Pi'(\theta^*) = -\frac{1+\eta\lambda}{1-\eta}\mathbb{E}[u'(\widetilde{w_0} + \theta^* Y)\,Y] < 0$ that there exists at least one $\xi \in (0,\theta^*)$ such that $\Pi'(\xi) = 0$, that is,

$$\left(1 - \frac{\eta\,(1-\lambda)}{1-\eta}\right)\Theta_+(\xi) = -\left(1 + \frac{\eta\,(1-\lambda)}{1-\eta}\right)\Theta_-(\xi)\,.$$

Combining this identity with $\Theta_-(\xi) < \Theta_-(0) \leq 0$ and $\Theta_+(\xi) > \Theta_+(\theta^*) > 0$ gives $\frac{\eta(1-\lambda)}{1-\eta} < 1$, so $\Pi'(\cdot)$ is strictly decreasing on $(0,+\infty)$ and $\Pi'(0+) > 0$. Hence, (A.16) implies $\Pi(\theta^*) < \Pi(0)$. □

Proof of Theorem 3.7. Let $\mu > 0$ (the case $\mu < 0$ is similar, so we omit the details). First, note that $0 < \theta^*(\eta,\lambda) < \theta$ for all suitable $\eta, \lambda \in (0,1)$ by the proof of Theorem 3.4, therefore the mapping $(\eta,\lambda) \mapsto \theta^*(\eta,\lambda)$ is bounded. We break the rest of the proof into several parts.

(i) *(a)* To show that $(\eta,\lambda) \mapsto \theta^*(\eta,\lambda)$ is continuous, fix an arbitrary (η,λ) for which (3.5) holds, and let $\{(\eta_n,\lambda_n)\}_{n\in\mathbb{N}}$ be a sequence converging to (η,λ). Without loss of generality, η_n and λ_n satisfy (3.5) for all $n \in \mathbb{N}$. Next, consider any convergent subsequence $\{\theta^*(\eta_{n_k},\lambda_{n_k})\}_{k\in\mathbb{N}}$ of the bounded sequence $\{\theta^*(\eta_n,\lambda_n)\}_{n\in\mathbb{N}}$. By continuity of $\Theta_\pm(\cdot)$,

$$\begin{aligned}\frac{1}{1-\eta}\Theta_-\Big(\lim_{k\to+\infty}\theta^*(\eta_{n_k},\lambda_{n_k})\Big) + \frac{1-\eta\,(1-\lambda)}{1-\eta}\Theta_+\Big(\lim_{k\to+\infty}\theta^*(\eta_{n_k},\lambda_{n_k})\Big)&\\ = \lim_{k\to+\infty}\Lambda(\eta_{n_k},\lambda_{n_k};\theta^*(\eta_{n_k},\lambda_{n_k})) = 0,&\end{aligned}$$

therefore $\lim_{k\to+\infty}\theta^*(\eta_{n_k},\lambda_{n_k}) > 0$ (otherwise $\Theta_-(0) + (1-\eta\,(1-\lambda))\,\Theta_+(0) = 0$, that is, $1 - \eta\,(1-\lambda) = \frac{G-1}{G+1}$). Consequently, $\Lambda(\eta,\lambda;\lim_{k\to+\infty}\theta^*(\eta_{n_k},\lambda_{n_k})) = 0 = \Lambda(\eta,\lambda;\theta^*(\eta,\lambda))$, which in turn gives $\lim_{k\to+\infty}\theta^*(\eta_{n_k},\lambda_{n_k}) = \theta^*(\eta,\lambda)$.

(b) In this step, we investigate how the unique risky equilibrium varies with η and λ. We claim that $\theta^*(\eta,\lambda)$ is strictly decreasing in η, for any $\lambda \in (0,1)$ fixed. To see this, let $\eta_1, \eta_2 \in (0,1)$ such that $\eta_1 < \eta_2$ and (3.5) holds. Combining $\mathbb{E}[u'(\widetilde{w_0} + \theta^*(\eta_1,\lambda)\,Y)\,Y] = \eta_1\,(1-\lambda)\,\Theta_+(\theta^*(\eta_1,\lambda))$ with $\theta^*(\eta_1,\lambda) < \theta$ yields $\Theta_+(\theta^*(\eta_1,\lambda)) > 0$. Then,

$$\Lambda(\eta_2,\lambda;\theta^*(\eta_1,\lambda)) = \frac{(\eta_1-\eta_2)\,(1-\lambda)}{1-\eta_2}\Theta_+(\theta^*(\eta_1,\lambda)) < 0,$$

so $\theta^*(\eta_1,\lambda) > \theta^*(\eta_2,\lambda)$. Likewise, $\lambda \mapsto \theta^*(\eta,\lambda)$ is strictly increasing for any fixed $\eta \in (0,1)$.

(c) Next, we show that the risky equilibrium is close to the EUT-optimal portfolio when the gain-loss sensitivity is close to zero. Indeed, let $\lambda \in (0,1)$ be given, and consider a sequence $\{(\eta_n, \lambda_n)\}_{n\in\mathbb{N}}$ with limit $(0, \lambda)$. Without loss of generality, η_n and λ_n satisfy (3.5) for all $n \in \mathbb{N}$. If $\{\theta^*(\eta_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$ is a convergent subsequence, then

$$\begin{aligned}\mathbb{E}\Big[u'\Big(\widetilde{w_0} + \lim_{k\to+\infty} \theta^*(\eta_{n_k}, \lambda_{n_k})\, Y\Big)\, Y\Big] &= \Theta_-\Big(\lim_{k\to+\infty} \theta^*(\eta_{n_k}, \lambda_{n_k})\Big) + \Theta_+\Big(\lim_{k\to+\infty} \theta^*(\eta_{n_k}, \lambda_{n_k})\Big)\\ &= \lim_{k\to+\infty} \Lambda(\eta_{n_k}, \lambda_{n_k}; \theta^*(\eta_{n_k}, \lambda_{n_k})) = 0,\end{aligned}$$

and so $\lim_{k\to+\infty} \theta^*(\eta_{n_k}, \lambda_{n_k}) = \theta$. Analogously, $\lim_{\lambda\to 1} \theta^*(\eta, \lambda) = \theta$ for all η.

(d) Finally, we study the limiting behavior of the risky equilibrium when the gain-loss sensitivity and loss aversion are both large.

(i') Suppose that $G \le 1$. Let $\{(\eta_n, \lambda_n)\}_{n\in\mathbb{N}}$ be a sequence tending to $(1, 0)$, and consider any convergent subsequence $\{\theta^*(\eta_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$. Then $\lim_{k\to+\infty} \theta^*(\eta_{n_k}, \lambda_{n_k}) = \bar{\theta}$, where $\bar{\theta} \ge 0$ denotes the unique root of $\Theta_-(\cdot)$, because

$$\begin{aligned}&\Theta_-\Big(\lim_{k\to+\infty} \theta^*(\eta_{n_k}, \lambda_{n_k})\Big) + (1-1)\,\Theta_+\Big(\lim_{k\to+\infty} \theta^*(\eta_{n_k}, \lambda_{n_k})\Big)\\ &\qquad = \lim_{k\to+\infty} (1-\eta_{n_k})\,\Lambda(\eta_{n_k}, \lambda_{n_k}; \theta^*(\eta_{n_k}, \lambda_{n_k})) = 0.\end{aligned}$$

Noticing that $\Theta_-(0)$ has the same sign as $1 - G$ gives $\bar{\theta} > 0$ if $G < 1$, and $\bar{\theta} = 0$ otherwise. Furthermore, $(1-\eta)\,\Lambda(\theta^*) = (1 - \eta\,(1-\lambda))\,\mathbb{E}[u'(\widetilde{w_0} + \theta^* Y)\, Y] + \eta\,(1-\lambda)\,\Theta_-(\theta^*)$ together with $\theta^*(\eta, \lambda) < \theta$ implies $\theta^*(\eta, \lambda) > \bar{\theta}$ for all $\eta, \lambda \in (0, 1)$.

(ii') Suppose that $G > 1$. Fix an arbitrary $\left(\bar{\eta}, \bar{\lambda}\right)$ satisfying (3.2). Let $\{(\eta_n, \lambda_n)\}_{n\in\mathbb{N}}$ be a sequence of vectors for which (3.5) holds and $\lim_{n\to+\infty} (\eta_n, \lambda_n) = \left(\bar{\eta}, \bar{\lambda}\right)$. Also, define the strictly decreasing function $\bar{\Lambda}\colon\ \mathbb{R} \to \mathbb{R}$ as

$$\bar{\Lambda}(z) := \Theta_-(z) + \frac{G-1}{G+1}\Theta_+(z)\,, \qquad \text{for all } z \in \mathbb{R}.$$

For any convergent subsequence $\{\theta^*\,(\eta_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$,

$$\begin{aligned}\bar{\Lambda}\Big(\lim_{k\to+\infty} \theta^*\,(\eta_{n_k}, \lambda_{n_k})\Big) &= \Theta_-\Big(\lim_{k\to+\infty} \theta^*\,(\eta_{n_k}, \lambda_{n_k})\Big) + \frac{G-1}{G+1}\Theta_+\Big(\lim_{k\to+\infty} \theta^*\,(\eta_{n_k}, \lambda_{n_k})\Big)\\ &= \lim_{k\to+\infty} (1-\eta_{n_k})\,\Lambda(\eta_{n_k}, \lambda_{n_k}; \theta^*(\eta_{n_k}, \lambda_{n_k})) = 0 = \bar{\Lambda}(0)\,,\end{aligned}$$

hence $\lim_{k\to+\infty} \theta^*\,(\eta_{n_k}, \lambda_{n_k}) = 0$.

(ii) Straightforward computations show that (3.8) implies $u_2'(x) = \rho'(u(x))\,u'(x)$ for all $x \in \mathbb{R}$. This identity, together with the monotonicity of $u_1(\cdot)$ as well as the concavity of $\rho(\cdot)$, leads to

$$u_2'\Big(\widetilde{w_0} + \theta_1^* Y\Big)\, Y = \rho'\Big(u_1\Big(\widetilde{w_0} + \theta_1^* Y\Big)\Big)\, u_1'\Big(\widetilde{w_0} + \theta_1^* Y\Big)\, Y \le \rho'(u_1(\widetilde{w_0}))\, u_1'\Big(\widetilde{w_0} + \theta_1^* Y\Big)\, Y.$$

As a consequence,

$$\begin{aligned}\Lambda_2\Big(\theta_1^*\Big) &\le \frac{1}{1-\eta}\mathbb{E}_{\mathbb{P}}\Big[\rho'(u_1(\widetilde{w_0}))\, u_1'\Big(\widetilde{w_0} + \theta_1^* Y\Big)\, Y\, \mathbb{1}_{\{Y<X\}}\Big] + \frac{1-\eta\,(1-\lambda)}{1-\eta}\mathbb{E}_{\mathbb{P}}\Big[\rho'(u_1(\widetilde{w_0}))\, u_1'\Big(\widetilde{w_0} + \theta_1^* Y\Big)\, Y\, \mathbb{1}_{\{Y>X\}}\Big]\\ &= \rho'(u_1(\widetilde{w_0}))\,\Lambda_1\Big(\theta_1^*\Big) = 0 = \Lambda_2\Big(\theta_2^*\Big),\end{aligned}$$

where $\Lambda_i(\cdot)$, $i \in \{1, 2\}$, is the function given by (A.10) with associated utility $u_i(\cdot)$.

(iii) The function defined by

$$G(w_0, z) := \mathbb{E}[u'(\widetilde{w_0} + zY)\, Y\, \mathbb{1}_{\{Y-X<0\}}] + (1 - \eta\,(1-\lambda))\, \mathbb{E}[u'(\widetilde{w_0} + zY)\, Y\, \mathbb{1}_{\{Y-X>0\}}]\,,$$

for all $(w_0, z) \in \mathbb{R} \times (0, +\infty)$, is continuously differentiable with[12]

$$\frac{\partial G}{\partial z} = \mathbb{E}\Big[u''(\widetilde{w_0} + zY)\, Y^2\, \mathbb{1}_{\{Y-X<0\}}\Big] + (1 - \eta\,(1-\lambda))\, \mathbb{E}\Big[u'(\widetilde{w_0} + zY)\, Y^2\, \mathbb{1}_{\{Y-X>0\}}\Big] < 0.$$

Thus, by the Implicit Function Theorem, the partial derivative $\partial\theta^* / \partial w_0$ has the same sign as $\partial G / \partial w_0$. Note also that

$$\begin{aligned}\frac{\partial G}{\partial w_0} = &-(1+r)\, \mathbb{E}[\mathrm{ARA}_u(\widetilde{w_0} + zY)\, u'(\widetilde{w_0} + zY)\, Y\, \mathbb{1}_{\{Y-X<0\}}] \\ &- (1 - \eta\,(1-\lambda))\,(1+r)\, \mathbb{E}[\mathrm{ARA}_u(\widetilde{w_0} + zY)\, u'(\widetilde{w_0} + zY)\, Y\, \mathbb{1}_{\{Y-X>0\}}]\,,\end{aligned}$$

and $\mathrm{ARA}_u(\widetilde{w_0} + zY)\, Y \le \mathrm{ARA}_u(\widetilde{w_0})\, Y$ (respectively, $\mathrm{ARA}_u(\widetilde{w_0} + zY)\, Y = \mathrm{ARA}_u(\widetilde{w_0})\, Y$ or $\mathrm{ARA}_u(\widetilde{w_0} + zY)\, Y \ge \mathrm{ARA}_u(\widetilde{w_0})\, Y$) if $u(\cdot)$ displays non-increasing absolute risk aversion (respectively, constant absolute risk aversion or non-decreasing absolute risk aversion). Recalling the first-order condition

$$\frac{1}{1-\eta}\mathbb{E}[u'(\widetilde{w_0} + \theta^* Y)\, Y\, \mathbb{1}_{\{Y-X<0\}}] + \frac{1 - \eta\,(1-\lambda)}{1-\eta}\mathbb{E}[u'(\widetilde{w_0} + \theta^* Y)\, Y\, \mathbb{1}_{\{Y-X>0\}}] = 0$$

gives the intended conclusion. □

Proof of Lemma 3.10. We deal with the case where $\mu > 0$, the other one being identical. We proceed in two separate steps.

(i) Suppose first that $u(\cdot)$ is linear. It is immediate to see that (3.9) implies the linearity of $\nu_-(\cdot)$ as well, thus in particular $\nu_-'(0) = \nu_-'(+\infty)$. As a consequence, $\lambda_\infty = \nu_+'(+\infty)\,/\nu_-'(0) \le \lambda$ (with equality if and only if $\nu_+'(\cdot)$ is linear).

Set $\hat{Y} := \mathbb{E}[Y|\, Y - X]$, and let $k > 0$. Using the measurability of $\mathbb{1}_{\{Y-X>k\}}$ with respect to the σ-algebra generated by the random variable $Y - X$, the definition of conditional expectation, and the independence of X and Y,

$$\mathbb{E}\Big[\hat{Y}\, \mathbb{1}_{\{Y-X>k\}}\Big] = \mathbb{E}[Y\, \mathbb{1}_{\{Y-X>k\}}] = \mathbb{E}[Y F(Y-k)] \ge \mu F(-k) \ge 0.$$

Since $k > 0$ is arbitrary, we see that

$$\hat{Y} \ge 0 \text{ a.s. on } \{Y - X > 0\}\,. \tag{A.17}$$

This in turn implies that the function $\Lambda(\cdot)$ defined in Lemma A.4 is non-increasing on $(0, +\infty)$, because for all $0 < z_1 < z_2$,

$$\begin{aligned}\Lambda(z_1) &= u'(\widetilde{w_0})\,(1 + \nu_-'(0))\, \mathbb{E}[Y\, \mathbb{1}_{\{Y-X<0\}}] + u'(\widetilde{w_0})\, \mathbb{E}\Big[(1 + \nu_+'(u'(\widetilde{w_0})\, z_1\, (Y-X)))\, \hat{Y}\, \mathbb{1}_{\{Y-X>0\}}\Big] \\ &\ge u'(\widetilde{w_0})\,(1 + \nu_-'(0))\, \mathbb{E}[Y\, \mathbb{1}_{\{Y-X<0\}}] + u'(\widetilde{w_0})\, \mathbb{E}\Big[(1 + \nu_+'(u'(\widetilde{w_0})\, z_2\, (Y-X)))\, \hat{Y}\, \mathbb{1}_{\{Y-X>0\}}\Big] \\ &= \Lambda(z_2)\,,\end{aligned}$$

where the first and last equalities follow from the tower property, while the inequality is a consequence of the concavity of $\nu_+(\cdot)$ combined with (A.17).

Furthermore, note that $\Lambda(0+) > 0$ and $\Lambda(+\infty) < 0$ are equivalent to (3.12) and

$$1 - \frac{\nu_-(0)\,(1-\lambda_\infty)}{1 + \nu_-(0)} < \frac{G-1}{G+1},$$

respectively. As a consequence, there are only three possible scenarios to consider.

[12] This fact requires an additional integrability condition involving $u''(\cdot)$ and X^2.

(a) If (3.10) holds (which can only occur for $v_+(\cdot)$ not linear), then the continuous function $\Lambda(\cdot)$ must cross the horizontal axis at least once on $(0, +\infty)$. On the other hand, since $\Lambda(\cdot)$ may not be strictly monotonic on the positive half-line, we cannot conclude anything about the number of its zeros. What we can say, however, is that due to $\Lambda(+\infty) < 0$ we must have $\Lambda(z) < 0$ for all z sufficiently large; moreover, if the number of roots of $\Lambda(\cdot)$ is finite, then it must be odd (otherwise $\Lambda(+\infty) > 0$, a contradiction).

(b) If (3.11) holds, then either $\Lambda(+\infty) > 0$ or $\Lambda(0+) < 0$, whence by monotonicity $\Lambda(\cdot)$ is always strictly above or strictly below zero.

(c) The case where either

$$\frac{G-1}{G+1} = 1 - \frac{v_-(0)\,(1-\lambda_\infty)}{1+v_-(0)} \leq 1 - \frac{v_-(0)\,(1-\lambda)}{1+v_-(0)}$$

or

$$\frac{G-1}{G+1} = 1 - \frac{v_-(0)\,(1-\lambda)}{1+v_-(0)} \geq 1 - \frac{v_-(0)\,(1-\lambda_\infty)}{1+v_-(0)}$$

is inconclusive whenever $v_+(\cdot)$ is non-linear.

(ii) As seen above, condition (3.12) implies $\Lambda(0+) > 0$. This inequality together with continuity and $\Lambda(+\infty) = -\infty$ entails the existence of at least one root for $\Lambda(\cdot)$ on $(0, +\infty)$. The remaining conclusions follow as before. □

Proof of Lemma 3.11. The proof comprises two steps.

(i) Let ψ' be an $\mathscr{F}_0'$-measurable random variable that is independent of, and has the same distribution as ψ. Since X and Y are also independent and identically distributed, $\mathbb{P}\{\psi' Y - \psi X > 0\} = \mathbb{P}\{\psi' Y - \psi X < 0\}$; moreover, $\mathbb{P}\{\psi' Y = \psi X\} = \mathbb{P}\{\psi = 0\}\,\mathbb{P}\{\psi' = 0\} < 1$ because X has continuous law and ψ is non-degenerate. These two observations imply that the events $\{\psi' Y - \psi X > 0\}$ and $\{\psi' Y - \psi X < 0\}$ both occur with strictly positive probability.

Next, defining $\Psi_1(x, y, z) := \mathbb{E}[u(\widetilde{w_0} + \psi' y) + v(u(\widetilde{w_0} + \psi' y) - u(\widetilde{w_0} + zx))]$,

$$\begin{aligned}
&\mathbb{E}[u(\widetilde{w_0} + \psi' Y) + v(u(\widetilde{w_0} + \psi' Y) - u(\widetilde{w_0} + \psi X))] \\
&\qquad = \mathbb{E}\Big[\mathbb{E}\Big[u(\widetilde{w_0} + \psi' Y) + v(u(\widetilde{w_0} + \psi' Y) - u(\widetilde{w_0} + \psi X))\Big| X, Y, \psi\Big]\Big] = \mathbb{E}[\Psi_1(X, Y, \psi)]\,,
\end{aligned}$$

where we use the tower property of conditional expectation to obtain the first equality, and the second one is a consequence of ψ' being independent of X, Y and ψ.

By Jensen's inequality (recall Assumption 3.9, which ensures the global concavity of the mapping $\psi' \mapsto u(\widetilde{w_0} + \psi' y) + v(u(\widetilde{w_0} + \psi' y) - u(\widetilde{w_0} + zx))$),

$$\Psi_1(x, y, z) \leq \mathbb{E}[u(\widetilde{w_0} + \bar{\psi} y) + v(u(\widetilde{w_0} + \bar{\psi} y) - u(\widetilde{w_0} + zx))]\,.$$

Note that the equality above is never attained for $u(\cdot)$ strictly concave; when $u(\cdot)$ is linear, the equality holds if and only if $\mathbb{P}\{\psi' y - zx > 0\} = 0$ or $\mathbb{P}\{\psi' y - zx < 0\} = 0$.

Hence,

$$\mathbb{E}[u(\widetilde{w_0} + \psi' Y) + v(u(\widetilde{w_0} + \psi' Y) - u(\widetilde{w_0} + \psi X))] \\ < \mathbb{E}[u(\widetilde{w_0} + \bar{\psi} Y) + v(u(\widetilde{w_0} + \bar{\psi} Y) - u(\widetilde{w_0} + \psi X))]$$

(the strict inequality being due to the initial observation that $\mathbb{P}\{\psi' Y - \psi X > 0\} = \mathbb{P}\{\psi' Y - \psi X < 0\} > 0$).

(ii) That the supremum on the left-hand side of (3.13) does not exceed the supremum on the right-hand side is trivial, since the former is taken over the smaller set $\mathbb{R} \subseteq \mathscr{R}$. To prove the reverse inequality, let $\psi \in \mathscr{R}$ and observe as in the previous step that

$$U\Big(V_T^{\psi} \Big| V_T^{\phi} \Big) = \mathbb{E}[\Psi_2(X, Y)] \,,$$

where $\Psi_2(x, y) := \mathbb{E}[u(\widetilde{w_0} + \psi y) + v(u(\widetilde{w_0} + \psi y) - u(\widetilde{w_0} + \phi x))]$ (recall that ϕ is a constant). Thus, setting $\bar{\psi} := \mathbb{E}[\psi] \in \mathbb{R}$, another application of Jensen's inequality yields

$$\Psi_2(x, y) \leq u(\widetilde{w_0} + \bar{\psi} y) + v(u(\widetilde{w_0} + \bar{\psi} y) - u(\widetilde{w_0} + \phi x))$$

and consequently $U\Big(V_T^{\psi} \Big| V_T^{\phi} \Big) \leq U\Big(V_T^{\bar{\psi}} \Big| V_T^{\phi} \Big) \leq \sup_{\varphi \in \mathbb{R}} U\Big(V_T^{\varphi} \Big| V_T^{\phi} \Big)$. □

Proof of Proposition 4.2. The proof consists of finding explicit expressions for the functions defined in Lemmata A.2 and A.4.

(i) We have

$$\Xi_0(z) = \begin{cases} \frac{1}{\gamma} + \frac{1}{\gamma} e^{-\gamma \widetilde{w_0}} \left(\frac{\eta \lambda}{1-\eta} + \frac{\eta(1-\lambda)}{1-\eta} \Phi\Big(\frac{\mu}{\sigma}\Big) \right) \\ \quad - \frac{1}{\gamma} e^{-\gamma \widetilde{w_0} - \gamma \mu z + \frac{1}{2}\gamma^2\sigma^2 z^2} \left(1 + \frac{\eta\lambda}{1-\eta} + \frac{\eta(1-\lambda)}{1-\eta} \Phi\Big(\frac{\mu}{\sigma} - \gamma\sigma z\Big) \right), & \text{if } z < 0, \\ \frac{1}{\gamma} + \frac{1}{\gamma} e^{-\gamma \widetilde{w_0}} \left(\frac{\eta}{1-\eta} - \frac{\eta(1-\lambda)}{1-\eta} \Phi\Big(\frac{\mu}{\sigma}\Big) \right) \\ \quad - \frac{1}{\gamma} e^{-\gamma \widetilde{w_0} - \gamma \mu z + \frac{1}{2}\gamma^2\sigma^2 z^2} \left(1 + \frac{\eta}{1-\eta} - \frac{\eta(1-\lambda)}{1-\eta} \Phi\Big(\frac{\mu}{\sigma} - \gamma\sigma z\Big) \right), & \text{if } z \geq 0. \end{cases}$$

(ii) Let $b \in \mathbb{R} \setminus \{0\}$. For all $z \in \mathbb{R}$,

$$\Xi_b(z) = \frac{1}{\gamma} + \frac{1}{\gamma} e^{-\gamma \widetilde{w_0} - \gamma \mu b + \frac{1}{2}\gamma^2\sigma^2 b^2} \left(\frac{\eta}{1-\eta} - \frac{\eta\,(1-\lambda)}{1-\eta} \Phi\left(\frac{(z-b)\,\mu + \gamma\sigma^2 b^2}{\sigma\sqrt{z^2+b^2}} \right) \right) \\ - \frac{1}{\gamma} e^{-\gamma \widetilde{w_0} - \gamma \mu z + \frac{1}{2}\gamma^2\sigma^2 z^2} \left(1 + \frac{\eta\lambda}{1-\eta} + \frac{\eta\,(1-\lambda)}{1-\eta} \Phi\left(\frac{\gamma\sigma^2 z^2 - (z-b)\,\mu}{\sigma\sqrt{z^2+b^2}} \right) \right).$$

(iii) For all $z \in \mathbb{R}$,

$$\Pi(z) = \frac{1}{\gamma} - \frac{1}{\gamma} e^{-\gamma \widetilde{w_0} - \gamma \mu z + \frac{1}{2}\gamma^2\sigma^2 z^2} \left(1 - \frac{\eta\,(1-\lambda)}{1-\eta} + 2\frac{\eta\,(1-\lambda)}{1-\eta} \Phi\left(\frac{\gamma\sigma\,|z|}{\sqrt{2}} \right) \right).$$

(iv) For all $z \in \mathbb{R}$,

$$\Theta_-(z) = e^{-\gamma \widetilde{w_0} - \gamma \mu z + \frac{1}{2}\gamma^2\sigma^2 z^2} \left(\mu - \gamma\sigma^2 z \right) \Phi\left(\frac{\gamma\sigma z}{\sqrt{2}} \right) - e^{-\gamma \widetilde{w_0} - \gamma \mu z + \frac{1}{2}\gamma^2\sigma^2 z^2} \frac{\sigma}{2\sqrt{\pi}} e^{-\left(\frac{\gamma\sigma z}{2}\right)^2}.$$

□

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